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GREEN'S RELATIONS AND QUASI-IDEALS[1])

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In generalizing Croisot's Theory of Decompositions [2] we came upon the subclass of regular absorbent semigroups. It was shown ([2] Corollary (3.6)) that a semigroup belonged to this class precisely when it was the mutually annihilating sum of subsemigroups each of which was a completely 0-simple semigroup when considered separately. OTTO STEINFELD has also found equivalent conditions [7] for such a decomposition. His results were in part a culmination of his investigations of quasi-ideals in semigroups ([4], [5], [6]). It is our purpose now to reconcile both these approaches and to examine quasi-ideals from the standpoint of Green's relations.

We give an interesting characterization of absorbency in terms of a partial order on Green's equivalence classes (1.6). We then show that for an absorbent semigroup with 0 each $\mathscr{H}$-class union 0 is a quasi-ideal (1.8) (indeed, these are the only 0-minimal quasi-ideals (3.6)). For regular or commutative semigroups we show that the converse is true. It is an open question as to whether the converse is always true. Finally we give a direct proof of two theorems and corollaries closely related to one of Steinfeld's, using techniques involving Green's relations.

1. PRELIMINARIES

We begin with a brief summary of definitions and results from [2]. When S is a semigroup with 0 we will find it convenient for $X \subseteq S$ to denote the subset $X \cup \{0\}$ by X^0.

As usual (cf. [1] p. 47) we will let $\mathscr{L}$, $\mathscr{R}$, $\mathscr{H}$, $\mathscr{D}$ denote Green's equivalence relations, where for example, $a\mathscr{L}b$ whenever $Sa \cup \{a\} = Sb \cup \{b\}$. We will use the appropriate capital letter with a subscript to denote the equivaelnce class of an element under one of these relations; thus, for example, H_a denotes the $\mathscr{H}$-equivalence class of $a \in S$.

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(1.1) **Definition.** A semigroup S with 0 is said to be 2^0-*regular* whenever for each $x \in S$ either $x^2 = 0$ or $x = x^2ux^2$ for some $u \in S$. (This is a generalization of Croisot's (2,2)-regular semigroup.)

(1.2) **Theorem.** ([2] Theorem (2.5).) *If S is a semigroup with 0 then S is 2^0-regular if and only if $a^2 \in H_a^0$ for each $a \in S$.*

(1.3) **Definition.** A semigroup S with 0 is said to be *absorbent* whenever $ab \in \in (R_a \cap L_b)^0$ for each $a, b \in S$.

(1.4) Remark. If we set $b = a$ in (1.3) we can conclude immediately from (1.2) that an absorbent semigroup is 2^0-regular.

(1.5) **Definition.** We partially order the set of $\mathscr{R}$ and $\mathscr{L}$-classes of a semigroup with 0 in the usual fashion where, for example, $R_x \leqq R_y$ precisely when $R(x) = = \{x\} \cup xS \subseteqq \{y\} \cup yS = R(y)$. We will say that the set of $\mathscr{R}$-[$\mathscr{L}$-] classes of a semigroup is *trivially partially ordered* whenever $R_x \leqq R_y$ [$L_u \leqq L_v$] implies that either $x = 0$ or $x\mathscr{R}y$ [$u = 0$ or $u\mathscr{L}v$]. We will say that S has *trivial class order* whenever both the set of $\mathscr{R}$-classes and $\mathscr{L}$-classes are trivially partially ordered.

The following theorem gives an interesting characterization of absorbency:

(1.6) **Theorem.** *A semigroup S with 0 is absorbent if and only if S has trivial class order.*

Proof. Let S be an absorbent semigroup with 0. Suppose $R_x \leqq R_y$. If $x = 0$ then $R_x = \{0\}$; if $x = y$ then $R_x = R_y$, in either case we are done. Suppose then that $x \neq y$ and $x \neq 0 \neq y$. From the definition of $R_x \leqq R_y$ we have $xS^1 \subseteqq yS^1$ and this implies that $x = ys$ for some $s \in S$. Then by absorbency $x \in R_y \cap L_s$ and hence $x\mathscr{R}y$ and $R_x = R_y$. Dually one shows that $L_u \leqq L_v$ implies either $u = 0$ or $L_u = L_v$. Whence S has trivial class order.

Conversely suppose that S has trivial class order. Let $a, b \in S$. Clearly $R_{ab} \leqq R_a$ and $L_{ab} \leqq L_b$. Thus either $ab = 0$ or $R_{ab} = R_a$ and $L_{ab} = L_b$. Whence, in either case $ab \in (R_a \cap L_b)^0$ and it follows that S is absorbent.

(1.7) **Definition.** (cf. [5] p. 262). A subset Q of a semigroup S is said to be a *quasi-ideal* whenever $SQ \cap QS \subseteqq Q$.

(1.8) **Theorem.** *If S is an absorbent semigroup then each $\mathscr{H}$-class union $\{0\}$ is a quasi-ideal.*

Proof. Let H be an $\mathscr{H}$-class union $\{0\}$. Suppose $x \in HS \cap SH$. If $x \neq 0$ then $x = hs = s'h'$ for some $h, h' \in H$ and $s, s' \in S$. It follows that $R_x \leqq R_h$ and $L_x \leqq L_{h'}$. By the above theorem S has trivial class order so that either $x = 0$ or $x \neq 0$ and $R_x = R_h$ and $L_x = L_{h'}$. Since $h\mathscr{H}h'$ in the latter case, we have $x \in R_x \cap L_x = R_h \cap$

$\cap L_{h'} = H \setminus \{0\}$. In either case $x \in H$ and we can conclude $HS \cap SH \subseteqq H$ so that H is a quasi-ideal by definition.

(1.9) **Proposition.** *If S is a semigroup with 0 such that each $\mathscr{H}$-class union $\{0\}$ is a quasi-ideal then S is 2^0-regular.*

Proof. Let $x \in S$ and let $H = H_x^0$. Then $H^2 \subseteqq SH \cap HS \subseteqq H$ since H is a quasi-ideal. Thus either $x^2 = 0$ or $x^2 \in H$. It follows from (1.2) that S is 2^0-regular.

(1.10) **Definition.** (Cf. [6], p. 235.) A quasi-ideal Q of a semigroup S with 0 is said to be 0-*minimal* if there is no quasi-ideal P of S such that $\{0\} \subset P \subset Q$. (We will reserve $\subset$ for strict inclusion.)

(1.11) **Proposition.** (Cf. [6], Theorem 5). *If a non-zero $\mathscr{H}$-class union $\{0\}$ is a quasi-ideal then it is a 0-minimal quasi-ideal.*

Proof. Let H be a non-zero $\mathscr{H}$-class union $\{0\}$ which is a quasi-ideal of S. Suppose that G is a quasi-ideal of S and $0 \subset G \subseteqq H$. Now let $g \in G \setminus \{0\}$ and let $h \in H \setminus \{0\}$. Since $g \in H \setminus \{0\}$, an $\mathscr{H}$-class, if $g \neq h$ we can find an $r, s \in S$ such that $rg = gs = h$ (i.e., $g\mathscr{H}h$). Now $h \in Sg \cap gS \subseteqq SG \cap GS \subseteqq G$ which shows that $H \setminus \{0\} \subseteqq G$. Whence $G = H$ and H is, by definition, a 0-minimal quasi-ideal.

The following gives one an idea as to the extent of a quasi-ideal.

(1.12) **Proposition.** *Any quasi-ideal is the union of its $\mathscr{H}$-classes*; *i.e. if Q is a quasi-ideal then $Q = \bigcup_{q \in Q} H_q$.*

Proof. Let Q be a quasi-ideal and let $q \in Q$. Let $h \in H_q$. If $h \neq q$ there exist $r, s \in S$ such that $rq = qs = h$. Thus $h \in SQ \cap QS \subseteqq Q$. Hence $H_q \subseteqq Q$ and it follows that $Q = \bigcup_{q \in Q} H_q$.

2. A MUTUALLY ANNIHILATING SUM OF COMPLETELY 0-SIMPLE SEMIGROUPS

(2.1) **Definition.** Let $\{S_\alpha\}_{\alpha \in A}$ be a collection of distinct semigroups except for a common 0. Then the semigroup $S = \bigcup_{\alpha \in A} S_\alpha$ with $S_\alpha S_\beta = \{0\} = S_\beta S_\alpha$ for $\alpha \neq \beta$ is said to be the *mutually annihilating sum* of all the S_α.

One readily checks that the S_α are ideals in the above sum. It is then easy to verify that S is also the 0-direct union of its ideals, S_α, according to Steinfeld:

(2.2) **Definition.** ([7], p. 66.) A semigroup with 0 is said to be the 0-*direct union* of its ideals S_α $(\alpha \in A)$ if $S = \bigcup_{\alpha \in A} S_\alpha$ and $S_\alpha \cap \left(\bigcup_{\beta \neq \alpha} S_\beta\right) = \{0\}$.

(2.3) **Definition.** ([7], p. 66.) The quasi-ideals $Q_{\lambda\lambda'}(\lambda, \lambda' \in \Lambda)$ of a semigroup S with 0 are said to form a *complete system* whenever the following three conditions hold:

1. $Q_{\lambda\lambda'} = \{0\}$ or $Q_{\lambda\lambda'}$ is a 0-minimal quasi-ideal of S,
2. if $Q_{\lambda\lambda'} \neq \{0\}$ then it is of the form $e_\lambda S e_{\lambda'}$ where e_λ and $e_{\lambda'}$ are idempotents,
3. If $Q_{\lambda\lambda'} \neq \{0\}$ then $Q_{\lambda'\lambda} Q_{\lambda\lambda'} \neq \{0\}$.

(2.4) **Theorem.** *If S is a non-trivial semigroup with 0 then the following conditions are equivalent:*

1. *S is regular and the union of its 0-minimal left ideals.*
2. *S is the union of 0-minimal left ideals of the form Se_λ where $e_\lambda^2 = e_\lambda$.*
3. *S is the 0-direct union of two-sided ideals which are completely 0-simple subsemigroups of S.*
4. *S is the union of quasi-ideals which form a complete system.*
5. *S is regular and the union of its 0-minimal quasi-ideals.*
6. *S is regular and absorbent.*
7. *S is regular and each $\mathcal{H}$-class union 0 is a quasi-ideal.*

Proof. That (3) implies (6) follows immediately from [2] Corollary (3.6) after checking the equivalence of the definitions (2.1) and (2.2). (6) implies (7) by (1.8) and (7) implies (5) by (1.11) since any semigroup is the union of its $\mathcal{H}$-classes.

Steinfeld has shown that conditions (1)–(5) are equivalent in [7] Theorem 15 and thus the proof is complete.

We also refer the reader to [3] for further equivalent conditions.

3. SOME FURTHER RESULTS

(3.1) **Remark and Open Question.** We have seen (1.9) that if each $\mathcal{H}$-class union $\{0\}$ of a semigroup is a quasi-ideal that S is 2^0-regular. That the converse is false is seen by most examples of a semilattice with 0. However, we do know from (1.8) that absorbency implies each $\mathcal{H}$-class union $\{0\}$ is a quasi-ideal. We have seen (2.4) that under the overall assumption of regularity these last two are equivalent; whether this is always true is an open question. The following two propositions shed further light on this question.

(3.2) **Proposition.** *If each $\mathcal{H}$-class union $\{0\}$ of a semigroup S with 0 is an ideal then S is absorbent.*

Proof. Let $a, b \in S$. Then since H_a^0, H_b^0 are ideals by hypothesis, $ab \in H_a^0 H_b^0 \subseteqq (H_a \cap H_b)^0 \subseteqq (R_a \cap L_b)^0$ and S is thus absorbent.

(3.3) **Lemma.** *For commutative semigroups, quasi-ideals and ideals coincide.*

Proof. If S is a commutative semigroup then $SQ = QS$ for any subset Q of S. Thus if Q is a quasi-ideal we have $QS \cap SQ \subseteq Q$. Hence since $SQ = QS$ it follows that QS, $SQ \subseteq Q$ and Q is an ideal. Clearly any ideal is also a quasi-ideal.

(3.4) **Theorem.** *If S is a commutative semigroup with* 0 *then S is absorbent if and only if each $\mathscr{H}$-class union $\{0\}$ of S is a quasi-ideal.*

Proof. One implication is just (1.8).

Conversely, suppose each $\mathscr{H}$-class union $\{0\}$ of S is a quasi-ideal. Then since S is commutative each quasi-ideal is an ideal by (3.3). We now apply (3.2) to complete the proof.

We conclude this paper with several theorems and corollaries which we believe further illustrate the relationship between quasi-ideals and Green's relations.

(3.5) **Theorem.** ([5] Theorem 1.) *The intersection of a left ideal and a right ideal of a semigroup S is a quasi-ideal. Conversely, any quasi-ideal Q of S can be obtained as the intersection of a left ideal and a right ideal, viz.,* $Q = (Q \cup SQ) \cap \cap (Q \cup QS)$.

(3.6) **Theorem.** *Let S be a semigroup with* 0. *A quasi-ideal Q is* 0*-minimal if and only if Q is an $\mathscr{H}$-class union* $\{0\}$.

Proof. If a quasi-ideal is an $\mathscr{H}$-class union $\{0\}$ then it is 0-minimal by (1.11).

Conversely, suppose Q is a 0-minimal quasi-ideal. Let $a, b \in Q \setminus \{0\}$. Since $S^1a \cap \cap aS^1$ and $S^1b \cap bS^1$ are non-zero quasi-ideals by (3.5) which are contained in Q we must have $S^1a \cap aS^1 = Q = S^1b \cap bS^1$. But this clearly implies $a\mathscr{H}b$. Whence $Q = H_a^0$.

(3.7) **Corollary** (cf. [6] Theorem 4). *A* 0*-minimal quasi-ideal is either a group with adjoined* 0 *or a null subsemigroup.*

Proof. From the above theorem a 0-minimal quasi-ideal Q is an $\mathscr{H}$-class union $\{0\}$. Thus since $Q^2 \subseteq Q$ for a quasi-ideal it follows from [1] Theorem 2.16 that either $Q^2 = \{0\}$ or $Q^2 = Q$ and in that case Q is a group with adjoined 0.

We believe that the technique of the following theorem is of independent interest (cf. [6] Theorem 6).

(3.8) **Theorem.** *Let L be a* 0*-minimal left ideal of a semigroup S. Suppose* $0 \neq e = = e^2 \in L$. *Then* $eL = eSe = H_e^0$.

Proof. Since L is a 0-minimal left ideal and $e^2 = e \neq 0$ we have $\{0\} \subset Se \subseteq L$. Thus $Se = L$ and $eSe = eL \subseteq L$. By [1] Lemma 2.43, $L \setminus \{0\}$ is an $\mathscr{L}$-class. Hence if $0 \neq h \in eL$ we have $h\mathscr{L}e$. Thus there is an r' such that $r'h = e$. Let $r = er'e$. Since $h = eh = he$ we have $rh = er'eh = er'h = e^2 = e$. Now, however, $r \in \in eL \setminus \{0\}$ and so $r\mathscr{L}e$. But $rh = e$ and $er = r$ imply $r\mathscr{R}e$. Hence $e\mathscr{H}r$.

Now $(hr)\, h = h(rh) = he = h$ and $h(r) = hr$ imply $hr\mathscr{R}h$. Since both h and $hr \in L \setminus \{0\}$, $h\mathscr{L}hr$. Hence $hr\mathscr{H}h$. But $(hr)^2 = h(rh)\, r = (he)\, r = hr$ so that H_h is a group by [1] Theorem 2.16. But now $h = eh \in R_e \cap L_h = R_e \cap L_e = H_e$ since $L_e \cap R_h = L_h \cap R_h = H_h$ is a group (cf. [1] Theorem 2.17). Whence $eL = H_e^0$.

(3.9) **Corollary.** *If L and e are as in* (3.8) *then eL is a* 0*-minimal quasi-ideal.*

(3.10) **Corollary.** *If L and e are as in* (3.8) *then $eL = eSe = H_e^0 = Sh \cap hS$ for any $h \in eSe \setminus \{0\}$.*

As a partial converse to (3.10) we have the following proposition:

(3.11) **Proposition.** *If $e^2 = e \neq 0$ and eSe is a* 0*-minimal quasi-ideal of a semigroup S then $eSe = H_e^0 = Sh \cap hS$ for any $h \in eSe \setminus \{0\}$.*

Proof. Let $h \in eSe \setminus \{0\}$. Thus $h = eh = he$ and hence $Sh \cap hS \subseteq S(eSe) \cap \cap (eSe)\, S \subseteq eSe$ is a non-trivial quasi-ideal contained in eSe. We must have therefore $Sh \cap hS = eSe$. Since $e \in eSe$ there are $r, s \in S$ such that $rh = hs = e$. These equations and $h = eh = he$ imply that $h\mathscr{H}e$ so that $eSe = H_e^0$.

(3.12) **Corollary.** *If $0 \neq e^2 = e \in Q$ a* 0*-minimal quasi-ideal then $Q = eSe = H_e^0$.*

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