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VALUATIONS IN GROUPS AND RINGS

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(Received December 8, 1967)

1. In [1], the author considered the valuations of groups and rings as mappings of these systems into semilattices and lattices respectively. As a matter of fact, this idea of valuation introduced by him, generalises the concept of the term used previously. There, the author established a connection between the valuations of groups and rings and the homomorphisms of the lattice $L(G)$ of subgroups of $G$ and the lattice $L(R)$ of ideals of $R$, into the valuation semilattice and lattice respectively. Here, the results of [1] have been strengthened and as such, the said connection can be given in a more explicit form.

2. A mapping $N : G \to P$, of an additive group $G$ into an upper semilattice $P$, is called valuation, if and only if, $N(a + b) \subseteq N(a) \cup N(b)$.

The valuation $N$ is called symmetric, if $N(a) = N(-a)$, for all $a \in G$. Let the lattice of subgroups of $G$ be denoted by $L(G)$. Let $V(G)$ be the set of all symmetric valuations of $G$ into $P$ and $HL(G)$, the set of all homomorphisms of the upper semilattice of the lattice $L(G)$ into $P$.

**Theorem 1.** Let $G$ be an additive abelian group and $P$ a complete upper semilattice. Then the sets $V(G)$ and $HL(G)$ have the same power.

**Proof.** Let $N \in V(G)$. If for any subgroup $G_1 \subseteq G$, we put $n(G_1) = \bigcup_{a \in G_1} N(a)$, then as it has been shown in [1], $n \in HL(G)$.

On the other hand, if $n \in HL(G)$, then by defining $N'(a) = n(\{a\})$, where $\{a\}$ is the cyclic group generated by $a \in G$, it has been shown in [1] that $N' \in V(G)$.

We shall show that this correspondence between $V(G)$ and $HL(G)$ is biunique.

Since $N$ is a symmetric valuation into the complete upper semilattice $P$, we have,$N'(a) = n(\{a\}) = \bigcup_{k=-\infty}^{\infty} N(ka) \subseteq N(a)$. On the other hand, $N(a) \subseteq \bigcup_{k=-\infty}^{\infty} N(ka) = n(\{a\}) = N'(a)$. Hence $N = N'$. 
Let now, \( n \in HL(G) \), \( N(a) = n(\{a\}) \), and \( n'(H) = \bigcup_{a \in H} N(a) \), where \( H \) is any subgroup of \( G \). Then \( n'(H) = \bigcup_{a \in H} N(a) = \bigcup_{a \in H} n(\{a\}) = n(\bigcup_{a \in H} \{a\}) = n(H) \). That is, \( n' = n \).

Hence the theorem.

3. Let \( R \) be a ring with 1. A mapping \( N : R \to L \), of the ring \( R \) into the lattice \( L \), is called valuation, if and only if,

i) \( N(ab) \subseteq N(a) \cap N(b) \),

ii) \( N(a + b) \subseteq N(a) \cup N(b) \).

As shown in [1], \( N \) is always symmetric. Let \( L(R) \) be the lattice of ideals of \( R \), \( V(R) \) be the set of all valuations of \( R \) into \( L \) and \( HL(R) \), the set of all homomorphisms of the upper semilattice of \( L(R) \) into the upper semilattice of \( L \).

**Theorem 2.** Let \( R \) be a commutative ring with 1 and \( L \) be a complete lattice. Then the two sets \( V(R) \) and \( HL(R) \) have the same power.

**Proof.** Let \( N \in V(R) \). If we define \( n(J) = \bigcup_{a \in J} N(a) \), where \( J \) is any ideal of \( R \), then as shown in [1], \( n \in HL(R) \).

Conversely, if \( n \in HL(R) \), then by putting \( N'(a) = n((a)) \), where \( (a) \) is the principal ideal generated by \( a \in R \), it has been proved in [1], that \( N' \in V(R) \).

We shall now show that this correspondence between \( V(R) \) and \( HL(R) \) is biunique. We have \( N'(a) = n((a)) = \bigcup_{q \in R} N(aq) \subseteq N(a) \).

On the other hand, \( N(a) \subseteq \bigcup_{q \in R} N(aq) = n((a)) = N'(a) \), since \( R \) contains 1. Consequently, \( N = N' \).

Further, let \( n \in HL(R) \), \( N(a) = n((a)) \), \( n'(J) = \bigcup_{a \in J} N(a) \), where \( J \) is any ideal in \( R \).

Then \( n'(J) = \bigcup_{a \in J} N(a) = \bigcup_{a \in J} n((a)) = n(\bigcup_{a \in J} (a)) = n(J) \).

That is, \( n' = n \).

Hence the theorem.

**References**


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