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ON A PROBLEM OF M. JIŘINA CONCERNING CONTINUOUS STATE BRANCHING PROCESSES

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1. Introduction. We recall the structure and relevant properties of the continuous state process, adopting the notation and terminology of [4]. The stochastic evolution of the process is governed by the "offspring" distribution (of a nonnegative variable), $F(x)$, per unit quantity of "parent"; the distribution is required to be infinitely divisible with Laplace transform $\Phi(s) = \exp\{-h(s)\}$, where $h(s)$ will be referred to as its "cumulant generating function" (c.g.f.). The fundamental assumption may then be expressed in the statement that, for $s \geqq 0$,

$$E[e^{-sX_{n+1}} \mid X_n, X_{n-1}, \ldots, X_0] = e^{-X_n h(s)} = [\Phi(s)]^{X_n} \tag{1.1}$$

The criticality parameter, m, is given by $m = \int_0^\infty x \, dF(x)$, $m \leqq \infty$. We shall exclude throughout the possibility that $F(x)$ is degenerate at a single point, so that in particular $m > 0$.

Unless otherwise stated, we shall also assume that the process is initiated by a unit quantity of "ancestor" ($X_0 \equiv 1$) in which case it follows from (1.1) that $h_n(s)$, the c.g.f. for the random variable X_n, is the n-th functional iterate of $h(s)$ so that $h_n(s) = h(h_{n-1}(s)) = h_{n-1}(h(s))$. (It is also easy to see that the distributions of X_n are themselves infinitely divisible).

In the course of a recent monograph [1], p. 33, M. JIŘINA has raised the following question: if the process $\{X_n\}$ is subcritical ($m < 1$), so that in fact $X_n \to 0$ strongly, i.e. with probability one, do there exist positive constants c_n such that the sequence $\{X_n/c_n\}$ approaches a non-degenerate random variable, in some sense? For the familiar Galton-Watson process, where the state space consists of the non-negative integers, the answer to this question is in the negative, for in this case the subcritical system is absorbed at zero, with probability one, after a finite number of steps i.e. $X_n = 0$, and hence $X_n/c_n = 0$, for sufficiently large n. A similar result holds in our present situation, when the distribution $F(x)$ has some mass at the origin (i.e. absorption is possible in a single step). However, if there is no concentration at the origin, Jiřina's conjecture can have an affirmative answer, and we shall show that in fact the necessary and sufficient condition for the existence of a suitable sequence of constants is that r, the first point of increase of $F(x)$, should be greater than zero. (The convergence of $\{X_n/c_n\}$ is in distribution.)

Using precisely the same technique, we show that for a supercritical process $(m > 1)$ there exists a sequence of constants c_n such that $\{X_n/c_n\}$ converges in distribution to a proper non-degenerate random variable if and only if $m < \infty$. This extends a result recently obtained by one of the authors for the GALTON-WATSON process (see [3]).

We conclude the section by noting the following property of the c.g.f. $h(s)$, on which the main proposition depends.

Lemma 1.1. *The function $h(s)/s$ is monotonic decreasing in $s > 0$. As $s \to \infty$, $h(s)/s \to r$, where $r < m$, and r is the first point of increase of $F(x)$.*

Proof. Since $h(s)$ is the c.g.f. of a proper distribution it is concave in $[0, \infty)$, and hence $h(s)/s$ is monotonic decreasing in this interval and hence approaches a limit $q \geqq 0$. Since $m = \lim_{s \to 0+} h(s)/s$, it follows $m \geqq q$, with $m = q$ if and only if $h(s) = ms$, a case which we have excluded.

Let r be the first point of increase of $F(x)$, $\xi > r$. Then

$$h(s) = -\log \int_{0-}^{\infty} e^{-sx}\, \mathrm{d}F(x) \leqq -\log \int_{0-}^{\xi+} e^{-sx}\, \mathrm{d}F(x) \leqq -\log\left(e^{-s\xi} F(\xi)\right) =$$

$$= s\xi - \log F(\xi)$$

where $F(\xi) > 0$, and hence $-\log F(\xi) < \infty$, from the assumption $\xi > r$. Thus

$$\lim_{s \to \infty} h(s)/s \leqq \xi$$

and so $q \leqq \xi$. Since $\xi > r$ is arbitrary, $q \leqq r$. On the other hand for $s > 0$,

$$h(s)/s = -\log\left\{\int_{r-}^{\infty} e^{-sx}\, \mathrm{d}F(x)\right\}/s \geqq -\log\left(e^{-sr}\right)/s = r$$

whence $q \geqq r$. Hence $q = r$.

2. The Subcritical Process. Our main results are contained in the following propositions concerning the subclass of subcritical processes of the kind described above (which we shall refer to as Jiřina processes). The superciritical case will be treated in § 3.

Theorem 2.1 *The necessary and sufficient condition for the existence of a sequence of positive constants $\{c_n\}$ such that $\{X_n/c_n\}$ converges in law to a proper, nondegenerate random variable is that $r > 0$.*

When this condition is satisfied, the constants are essentially unique, in that, if $\{c_n\}$, $\{c'_n\}$ *are two sequences, then* $c_n \sim Kc'_n (0 < K < \infty)$ *as* $n \to \infty$. *The c.g.f.* $\gamma(s)$ *of the limit distribution satisfies the Poincaré functional equation*

$$h(\gamma(s)) = \gamma(rs) \quad s \geqq 0 \tag{2.1}$$

and is, apart from scale factors, the only c.g.f. of a proper, nondegenerate non-negative random variable to do so (i.e. the only other proper nondegenerate c.g.f. solutions are of the form $\gamma(s/c)$, *where c is a positive constant.)*

Proof. In view of the remarks of § 1, we may assume that $F(0) = 0$, i.e. there is no probability concentration at the origin.

Now let $k_n(t)$ be the inverse function to $h_n(s)$; under the original assumption that $F(x)$ is nondegenerate, this certainly exists. Under the further assumption that there is no concentration at zero, each $k_n(t)$ is in fact defined for all $t \in [0, \infty)$. We also note that $k_n(t)$ is the n-th functional iterate of $k(t) \equiv k_1(t)$, and that the inverse function to $h_n(s/c_n)$ (the c.g.f. of X_n/c_n) is $c_n\, k_n(t)$.

Next we introduce the functions $g_n(u)$, $n \geqq 1$, defined by

$$g_n(u) = 1/k_n(1/u)\,, \quad u > 0 \tag{2.2}$$

and note that $g_n(u)$ is the n-th functional iterate of $g_1(u) \equiv g(u)$, $u > 0$. Further since $\lim_{t\to\infty} k(t) = \infty$, we define $g(0)$ as 0 by continuity. It is now readily checked, that $g(u)$ has the following properties (following from those of $h(s)$, through $k(t)$)

(i) $g(u)$ is continuous and strictly increasing in $[0, \infty)$,
(ii) $g(0) = 0$ and $0 < g(u) < u$ for $u \in (0, \infty)$,
(iii) $\lim_{u\to 0+} g(u)/u = r$, $(0 \leqq r < 1)$,
(iv) $g(u)/u$ is monotone in $(0, \infty)$.

In particular property (iii) follows from Lemma 1.1, since

$$r = \lim_{s\to\infty} \frac{h(s)}{s} = \lim_{t\to\infty} \frac{t}{k(t)} = \lim_{u\to 0} \frac{g(u)}{u}\,.$$

If $r > 0$, a direct application of the theorem of Kuczma [2] yields that for a fixed $u_0 > 0$, and all $u > 0$

$$\frac{g_n(u)}{g_n(u_0)} \to \alpha(u)\,, \tag{2.3}$$

where $\alpha(u)$ is positive and is, up to constant factors, the unique solution, such that $\alpha(u)/u$ is monotonic in $(0, \infty)$, of the Schröder equation

$$\alpha(g(u)) = r\,\alpha(u)\,, \quad u > 0 \tag{2.4}$$

Putting $c_n = g_n(u_0) = k_n(t_0)^{-1}$ $(t_0 = 1/u_0)$, we have $c_n\, k_n(t) \to 1/\alpha(1/t) \equiv \beta(t)$ say, for $t > 0$, with $\beta(k(t)) = r^{-1} \beta(t)$. We may now imitate the proof of Theorem 3.1 of [3] (via preceding results of that paper) to deduce that in fact we may pass to the inverse functions, so that

$$h_n(s/c_n) \to \gamma(s)\,, \quad s \geqq 0$$

where $\gamma(s)$ is in fact concave, continuous, and strictly monotone increasing in $[0, \infty)$. Moreover $\gamma(s)$ satisfies

$$\gamma(rs) = h(\gamma(s))\,, \quad s \geqq 0$$

as required. Since in particular $\gamma(s)$ is continuous at the origin the continuity theorem for Laplace transforms yields the assertion of convergence in distribution to a proper random variable with $c_n = k_n(t_0)^{-1}$ (so that $c_n \to 0$ as $n \to \infty$). However (2.1) cannot have a solution of the form $\gamma(s) = \text{const.}\, s$, as this would imply a similar form for $h(s)$, and this case has been excluded. Thus the limit distribution is nondegenerate. Uniqueness follows from Kuczma's uniqueness assertion. Finally, it is not difficult to see, letting $s \to \infty$ in (2.1), that $h(\gamma(\infty)) = \gamma(\infty)$, whence $\gamma(\infty) = \infty$, i.e. the limit distribution has no mass at the origin.

Let us pass on now to the converse part of the theorem. Suppose a positive sequence $\{c_n\}$ is given, and that $\{X_n/c_n\}$ converges in law to a proper, nondegenerate random variable with c.g.f. $\gamma(s)$, so that $h_n(s/c_n) \to \gamma(s)$ for $s \geqq 0$. First note that there can be essentially only one such sequence. For suppose $\{\tilde{c}_n\}$ is another such sequence, $\tilde{\gamma}(s)$ the corresponding limit c.g.f. On taking inverse functions (appealing to Lemma 2.1 of [3]), we obtain

$$\left.\begin{aligned} c_n\, k_n(t) &\to \beta(t) > 0 \\ \tilde{c}_n\, \tilde{k}_n(t) &\to \tilde{\beta}(t) > 0 \end{aligned}\right\} t > 0$$

where $\beta(.)$ and $\tilde{\beta}(.)$ are respectively the inverses of $\gamma(.)$ and $\tilde{\gamma}(.)$. Hence $c_n/\tilde{c}_n \to K \equiv \equiv \beta(t)/\tilde{\beta}(t)$, so that the constants, as well as the limit function, are essentially unique.

We have, further, that

$$c_n\, k_n(k(t)) = [c_{n+1}\, k_{n+1}(t)] \left[\frac{c_n}{c_{n+1}}\right]$$

and since $c_n\, k_n(t) \to \beta(t)$ it follows, for $t > 0$, that $\mu = \lim_{n\to\infty} c_n/c_{n+1}$ exists, is equal to $\beta[k(t)]/\beta(t)$, and so satisfies $0 < \mu < \infty$. From $c_{n+1}\, k_{n+1}(t) \sim \beta(t) \sim c_n\, k_n(t)$ follows $c_n/c_{n+1} \sim k_{n+1}(t)/k_n(t)$, so that

$$\lim_{n\to\infty} \frac{k[k_n(t)]}{k_n(t)} = \mu\,.$$

But since $h(s)/s \to r$ as $s \to \infty$ (Lemma 1.1), $k(t)/t \to r^{-1}$ as $t \to \infty$. Thus $r^{-1} = \mu$ (note $k_n(t) \to \infty$ as $n \to \infty$), and so $r > 0$ from $\mu < \infty$. Hence the condition $r > 0$ is necessary as well as sufficient.

Corollary. *If* $r > 0$, $c_n \sim k_n(t_0)^{-1}$ $(n \to \infty)$ *for some fixed positive* t_0, *and the power series* $\sum c_n z^n$ *has radius of convergence* r^{-1}.

Theorem 2.2. *When* $r > 0$, $c_n \sim Kr^n$ *if and only if*

$$\int_a^\infty \frac{h(s) - rs}{s^2} \, \mathrm{d}s \tag{2.5}$$

converges (for any fixed $a > 0$*). (In particular (2.5) holds when* $F(r) > 0$ *i.e. there is a concentration of probability at the first point of increase).*

Proof. In order to prove the first assertion, we need to show that for a fixed $t_0 \in \in (0, \infty)$, $k_n(t_0) \sim K^{-1}r^{-n}$ when (2.5) holds, and conversely. This may be approached along the same lines as the corresponding result in [4]. Writing the quantity $k_n(t_0)\, r^n$ in the product form $\{t_0 \prod_{p=0}^{n-1} r\, k(k_p(t_0))/k_p(t_0)\}$ (where $k_0(s) = s$), and noting $r\, k(t)/t < < 1$, we see that the condition for convergence of the infinite product reduces to that for the convergence of $\sum\{r^{-1} - G(k_p(t_0))\}$ where $G(t) = k(t)/t$. Now it follows from the Corollary to Theorem 1.1 that $k_n(t_0)$ can be bounded below by a sequence of the form $A\theta^{-p}(A > 0,\ 0 < \theta < 1)$, and since $r\, k(t)/t < 1$, from above by the sequence r^{-p}. The integral test shows, finally, that the necessary and sufficient condition for the convergence of a series of the form $\sum\{r^{-1} - G(A\theta^{-p})\}$ is independent of A and θ in the specified ranges, and reduces to the convergence of the integral

$$\int_b^\infty \frac{r^{-1} - G(t)}{t} \, \mathrm{d}t\,, \quad (b > 0)\,. \tag{2.6}$$

Converting back to the c.g.f. $h(s)$ by putting $t = h(s)$ this becomes

$$\int_{k(b)}^\infty \frac{h(s) - rs}{r(h(s))^2} \, h'(s) \, \mathrm{d}s$$

and since $h'(s) \to r$ (this may easily be shown), and $h(s)/s \to r$ as $s \to \infty$, convergence or divergence of (2.6) is equivalent to that of

$$\int_a^\infty \frac{h(s) - rs}{s^2}\,, \quad (a > 0)$$

which proves the first assertion.

If $F(r) \equiv \delta > 0$, then

$$h(s) = -\log\left\{\int_{r-}^\infty e^{-sx} \, \mathrm{d}F(x)\right\} = sr - \log\delta - \log\left\{\int_{r+}^\infty e^{-sx} \, \mathrm{d}F(x)\right\}.$$

The integrand of (2.5) is then $(-\log\delta)/s^2 + o(s^{-2})$ as $s \to \infty$, and hence the integral converges.

In general, there seems to be no way of expressing the integral condition directly in terms of the distribution function $F(.)$. As suggested by the second part of Theorem 2.2, the convergence of the integral is a requirement to the effect that $F(x)$ should not be "too thinly" distributed in the neighbourhood of r. In terms of the representation of Lévy-Khinchin type

$$h(s) = rs + \int_{0+}^{\infty} (1 - e^{-sx})\varrho\{dx\}$$

(see for example the appendix of [4]), the integral (2.5) reduces to

$$\int_{0+}^{\infty} \left\{ \int_{a}^{\infty} \frac{1 - e^{-sx}}{s^2} \, ds \right\} \varrho\{dx\} = \int_{0+}^{\infty} \left\{ \int_{ax}^{\infty} \frac{1 - e^{-y}}{y^2} \, dy \right\} x \, \varrho\{dx\} .$$

The inner integral behaves as $\log x$ as $x \to 0$, and as x^{-1} as $x \to \infty$. Since ϱ allots finite mass to any complement of a neighbourhood of the origin, the outer integral always converges at infinity, and so finally (2.5) converges or diverges with

$$\int_{0+}^{\varepsilon} x \log x \, \varrho\{dx\} , \quad (\varepsilon > 0) . \tag{2.7}$$

(To see that this is reasonable, recall that $\varrho\{.\}$ always integrates x at the origin, and is totally finite if and only if there is a concentration of mass at r.) (2.7) is reminiscent of the result obtained in Theorem 3.2 below, and Theorem 3 of [4], where $F(x)$ appears in place of $\varrho\{.\}$.

To conclude this section we remark that the limit distribution described by Theorem 2.1 is always infinitely divisible, and that the distribution with c.g.f. $\theta\,\gamma(s)$ can be obtained by taking the same sequence of constants and initial mass θ. From this it also follows that the condition of Theorem 2.1 is valid even in the case of an arbitrary initial distribution; when it is satisfied the c.g.f. of the limit distribution has the form $c_0[\gamma(s)]$, where $c_0(s)$ is the c.g.f. of the initial mass, and $\gamma(s)$ is the solution characterized by Theorem 2.1.

3. The Supercritical Process. The following theorems parallel those of the preceding section, when $m > 1$.

Theorem 3.1. *The necessary and sufficient condition for the existence of a sequence of positive constants* $\{c_n\}$ *such that* $\{X_n/c_n\}$ *converges in law to a proper nondegenerate random variable is that* $m < \infty$.

When this condition is satisfied the constants are essentially unique, in that if $\{c_n\}, \{c_n'\}$ *are two such sequences, then* $c_n \sim Kc_n'$ $(0 < K < \infty)$ *as* $n \to \infty$. *The c.g.f.* $\gamma(s)$ *of the limit distribution satisfies the Poincaré functional equation*

$$h(\gamma(s)) = \gamma(ms) , \quad s \geqq 0 \tag{3.1}$$

and is, apart from scale factors, the unique c.g.f. of a proper, nondegenerate distribution of a non-negative random variable to do so.

Corollary. *If* $m < \infty$, $c_n \sim k_n(t_0)^{-1}$ $(n \to \infty)$ *for some fixed, positive* t_0, *and the radius of convergence of the series* $\sum_{n=0}^{\infty} c_n z^n$, *is* m^{-1}.

Theorem 3.2. *When* $m < \infty$, $c_n \sim Km^n$ *if and only if*

$$\int_0^\varepsilon \frac{h(s) - ms}{s^2} \, \mathrm{d}s \tag{3.2}$$

converges. The convergence or divergence of (3.2) *is equivalent to the convergence or divergence, respectively, of*

$$\int_0^\infty x \log x \, \mathrm{d}F(x) \equiv E[X_1 \log X_1] \,. \tag{3.3}$$

We give only a bare outline of the proofs, since in main they follow closely the proofs for the corresponding Galton-Watson process, given in [3].

Again, in the proof of Theorem 3.1, the key is the use of the inverse function iterates $k_n(.)$. However, since the point $t = 0$ is now an attractive fixpoint of $k(.)$, Kuczma's result must be applied directly to this function, without the further transformation to $g(.)$. The only new assertion in this theorem as compared to those of [3] is that $m < \infty$ is *necessary for the existence of the sequence of constants* $\{c_n\}$ (which turn out to be, as before, asymptotically proportional to the iterates $k_n(t_0)^{-1}$, for some fixed positive t_0). This assertion, and the remaining assertion of the Corollary, follow very much as in the proof of Theorem 2.1 of the preceding section, while the transition to the log moment condition (3.3) may be obtained in precisely the same way as in the argument of Theorem 3 of [4].

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