Hana Lovicarová
Periodic solutions of a weakly nonlinear wave equation in one dimension


Persistent URL: [http://dml.cz/dmlcz/100899](http://dml.cz/dmlcz/100899)

Terms of use:

© Institute of Mathematics AS CR, 1969

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.
PERIODIC SOLUTIONS OF A WEAKLY NONLINEAR WAVE EQUATION IN ONE DIMENSION

HANA LOVICAROVÁ, Praha

(Received April 18, 1968)

1. INTRODUCTION

In this paper we shall investigate the equation

\[(1.1) \quad u_{tt} - u_{xx} = \varepsilon f(t, x, u, u_t, u_x)\]

on the domain \(G = \mathbb{R} \times (0, \pi) (\mathbb{R} = (-\infty, \infty))\) of the plane \((t, x)\) with the boundary conditions

\[(1.2) \quad \lim_{(t, x) \to (t, 0)} u(t, x) = 0 = \lim_{(t, x) \to (t, \pi)} u(t, x).\]

We shall seek a solution of the problem (1.1), (1.2) 2\pi-periodic in \(t\) under the assumption that the function \(f\) is 2\pi-periodic in \(t\).

Vejvoda in [1] gave some sufficient conditions for the existence of periodic solutions of the problem (1.1), (1.2). In [3] the existence of 2\pi-periodic solution of the problem (1.1), (1.2) is proved if \(f\) depends only on \(t, x, u\) and \(f_u \leq -\gamma < 0\). Further in this paper Rabinowitz proved, that the problem (1.1), (1.2) has a 2\pi-periodic solution if the right hand side of the equation (1.1) has the form \(\varepsilon (a u_t + g(t, x, u))\) where \(a\) is a constant.

In this paper in paragraph 2 the existence of 2\pi-periodic solution in the linear case is treated. In paragraph 3 some auxiliary theorems are introduced. In paragraph 4 the existence of 2\pi-periodic solution of the problem (1.1), (1.2) is proved under the assumptions \(\partial f/\partial u_t \leq -\gamma < 0\) on \(G_1\) and \(\sup_{G_1} \partial f/\partial u_x - \inf_{G_1} \partial f/\partial u_x < \gamma\), where \(G_1 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}\), and certain restriction on the growth of \(f\). In paragraph 5 the existence and continuous dependence on \(\varepsilon\) of 2\pi-periodic solution, if \(f = f(t, x, u, \varepsilon)\) and \(f_u \leq -\gamma < 0\), is proved under weaker assumptions on the smoothness of \(f\) than in paper [3].

We conclude the introduction with some notations. Let \(D_i f\) denote the derivative of \(f\) with respect to \(i^{th}\) variable. Then we shall denote by \(C_k\) the Banach space of
functions defined on $G$, $2\pi$-periodic in $t$, for which $D^k_i D^j_x f (i + j \leq k, i, j \geq 0)$ are continuous and bounded on $G$, with the norm:

$$f \in C_k : \|f\|_{C_k} = \|f\|_k = \sup_{i+j \leq k} \sup_{(t,x) \in G} |D^k_i D^j_x f(t, x)|.$$ 

Let us note, that functions which belong to $C_k$ have derivatives up to the order $k - 1$ continuous up to the boundary of $G$.

Let $C^k$ denote the Banach space of $2\pi$-periodic functions $p$ (of one variable) which are continuous together with their derivatives up to the order $k$ and for which $[p] = \{0\}$. The norm in $C^k$ is given: $|p|_{C^k} = |p|_k = \sup |D^i p(y)|; 0 \leq i \leq k, y \in R$.

The space of all linear operators mapping a Banach space $B_1$ into a Banach space $B_2$ will be denoted by $[B_1 \to B_2]$.

$R(A), N(A)$ respectively denote a range and a null space respectively of the operator $A$.

2. THE LINEAR CASE

It is known (see e.g. [1]), that every classical solution of the homogeneous problem (1.1), (1.2) is $2\pi$-periodic and has a form

$$u(t, x) = p(t + x) - p(t - x),$$

where $p$ is $2\pi$-periodic and continuous together with its second derivative.

For nonhomogeneous equation

$$u_{tt} - u_{xx} = f(t, x)$$

we shall derive necessary and sufficient condition for the existence of $2\pi$-periodic solution, which fulfills the condition (1.2).

Let $u \in C^2$ be a solution of (2.2), (1.2). Integrating the equation (2.2) over the triangle $[t, x], (t - x + \pi, \pi), (t + x - \pi, \pi)$ and using the Green formula on the left hand side we obtain (as $u(t, 0) = 0)$:

$$u(t, x) = -\frac{1}{2} \int_{t-x+\pi}^{t+x-\pi} u_x(\tau, \pi) d\tau - \frac{1}{2} \int_{t-x-\pi}^{t+x+\pi} f(\tau, \xi) d\tau d\xi.$$ 

Since $u(t, 0) = 0$, we get

$$\int_{\pi}^{\pi} \int_{t-x-\pi}^{t+x+\pi} f(\tau, \xi) d\tau d\xi = -\int_{t-x-\pi}^{t+x+\pi} u_x(\tau, \pi) d\tau = \text{const}$$

because $u_x$ is also $2\pi$-periodic. Differentiating this relation with respect to $t$ we get

$$\int_{0}^{\pi} \left[ f(t + \xi, \xi) - f(t - \xi, \xi) \right] d\xi = 0.$$
If the function $f$ has continuous and bounded derivative $D_1 f$, the condition (2.3) is also sufficient for the existence of a solution $u \in C_2$ of the problem (2.2), (1.2). Indeed, if (2.3) holds, then
\begin{equation}
(2.4) \quad \int_0^\pi \int_{t-\xi}^{t+\xi} f(\tau, \xi) \, d\tau \, d\xi = \text{const} = k
\end{equation}
and it is easily seen, that the function
\begin{equation}
(2.5) \quad u(t, x) = -\frac{1}{2} \int_0^\pi \int_{t-x-\zeta}^{t-x+\zeta} f(\tau, \zeta) \, d\tau \, d\zeta + \frac{1}{2} \frac{\pi - x}{\pi} k
\end{equation}
is the sought solution.

Now let us prove, that the space $C_0$ can be written in the form of a direct summ $C_0 = N \oplus N^\perp$ where $N$ is the set of such functions for which (2.3) is fulfilled (i.e. for which there exists a solution of the problem (2.2), (1.2)) and $N^\perp$ is the set of functions which have the form (2.1) (i.e. which are solutions of the homogeneous problem).

Let us define the operators $Z$ and $Q$ on the spaces $C^0$ and $C_0$ respectively:
\begin{equation}
(2.6) \quad p \in C^0 : Zp(t, x) = p(t + x) - p(t - x) , \quad (t, x) \in G ,
\end{equation}
\begin{equation}
(2.7) \quad f \in C_0 : Qf(y) = \frac{1}{2\pi} \int_0^\pi (f(y - s, s) - f(y + s, s)) \, ds , \quad y \in R .
\end{equation}

**Lemma 2.1.** 1) $Z \in [C^k \to C_k]$ for $k \geq 0$ and
\begin{equation}
(2.8) \quad p \in C^k \Rightarrow \|Zp\|_k \leq 2|p|_k .
\end{equation}
2) $Q \in [C_k \to C^k]$ for $k \geq 0$ and
\begin{equation}
(2.9) \quad f \in C_k \Rightarrow \|Qf\|_k \leq \|f\|_k .
\end{equation}
3) $QZ = E$.
4) The operator $P_1 = ZQ$ is a projector of $C_0$ on $R(Z)$ and for $f \in C_k$ it holds
\begin{equation}
(2.10) \quad \|P_1 f\|_k \leq 2\|f\|_k .
\end{equation}
5) The operator $P_2 = E - P_1$ is a projector of $C_0$ on $N(Q)$ and for $f \in C_k$ it holds
\begin{equation}
(2.11) \quad \|P_2 f\|_k \leq 3\|f\|_k .
\end{equation}

**Proof.** It is obvious that 1) holds.

ad 2) Let $f \in C_0$. Evidently, $Qf$ is continuous and $2\pi$-periodic. Let us prove, that $[Qf] = 0$.
\begin{align*}
[Qf] &= \int_0^{2\pi} Qf(y) \, dy = \frac{1}{2\pi} \int_0^{2\pi} \left( \int_0^\pi (f(y - s, s) - f(y + s, s)) \, ds \right) \, dy = \\
&= \frac{1}{2\pi} \int_0^{2\pi} \left( \int_0^\pi f(y - s, s) \, dy - \int_0^\pi f(y + s, s) \, dy \right) \, ds = \emptyset
\end{align*}
because of \( \int_0^{2\pi} f(y - s, s) \, dy = \int_0^{2\pi} f(y + s, s) \, dy \). For \( f \in C_k \) it holds: \( D'Qf(y) = QD'_f(y) \) and from here we get, that \( Qf \in C_k \) and \( |Qf| \leq \|f\|_k \).

ad 3) Let \( p \in C^0 \). Then \( Zp(t, x) = p(t + x) - p(t - x) \) and so

\[
QZ \, p(y) = \frac{1}{2\pi} \int_0^{2\pi} \left[ (p(y) - p(y - 2s)) - (p(y + 2s) - p(y)) \right] \, ds =
\]

\[
= p(y) - \frac{1}{2\pi} \int_0^{2\pi} p(y - 2s) \, ds - \frac{1}{2\pi} \int_0^{2\pi} p(y + 2s) \, ds = p(y)
\]

because \( \int_0^{2\pi} p(y \pm 2s) \, ds = \frac{1}{2} [p] = 0 \).

ad 4) \( P_1 = (ZQ) \, (ZQ) = Z(QZ) \, Q = ZQ = P_1 \) and so \( P_1 \) is a projector. For \( f \in C_k \), \( P_1 \, f \in C_k \) and further \( \|P_1 \, f\|_k = \|Z(Qf)\|_k \leq 2 \|Qf\|_k \leq 2 \|f\|_k \). Obviously \( R(P_1) \subset R(Z) \). On the other hand if \( f \in R(Z) \), then there exists \( p \in C^0 \) such that \( f = Zp \) and \( P_1 \, f = (ZQ) \, Zp = Z(QZ) \, p = Zp = f \) and so \( R(P_1) = R(Z) \).

ad 5) According to 4) \( P_2 \) is a projector and for \( f \in C_k \) \( \|P_2 \, f\|_k = \|(E - P_1) \, f\|_k \leq \|f\|_k + \|P_1 \, f\|_k \leq 3 \|f\|_k \). Let us prove, that \( R(P_2) = N(Q) \). \( QP_2 = Q(E - QZ) = Q - QZQ = 0 \) and so \( R(P_2) \subset N(Q) \). For \( f \in N(Q) \): \( P_2 \, f = (E - QZ) \, f = f - Z(Qf) = f \) and so \( R(P_2) = N(Q) \).

Let us denote \( N = N(Q) = R(P_2), \, N^\perp = R(Z) = R(P_1) \). Then by lemma 2.1 \( C_0 = N \oplus N^\perp \). Further let \( N_k \) denote \( N \cap C_k \). From lemma 2.1 it follows easily, that \( N_k \) is a closed subspace of \( C_k \) and \( P_2 \) is a linear bounded operator from \( C_k \) onto \( N_k \).

Let us define the operators \( S' \) and \( S \) on the space \( N \) :

\[
(2.12) \quad S'(t, x) = \frac{1}{2\pi} \int_0^{2\pi} f(\tau, \xi) \, d\tau \, d\xi, \quad (t, x) \in G,
\]

\[
(2.13) \quad S(t, x) = -\frac{1}{2\pi} \int_0^{2\pi} f(\tau, \xi) \, d\tau \, d\xi, \quad (t, x) \in G,
\]

and let us prove the following

**Lemma 2.2.**

\[
(2.14) \quad 1) \quad S' \in [N_k \to C_{k+1}], \quad \|S'f\|_{k+1} \leq \frac{\pi^2}{2} \|f\|_k.
\]

\[
(2.15) \quad 2) \quad S \in [N_k \to C_{k+1}], \quad \|Sf\|_{k+1} \leq \left(k + \frac{\pi^2}{2}\right) \|f\|_k.
\]

**Proof.** ad 1) For \( f \in N \) \( \int_0^{2\pi} [t+x] f(\tau, \xi) \, d\tau \, d\xi = \text{const} \) and so only \( S'f \) and \( D_2 S'f \) are different from zero.

\[
\|S'f\|_0 \leq \frac{1}{2\pi} \|f\|_0 \left(\int_0^{2\pi} f(t, \xi) \, d\tau \, d\xi\right) = \frac{\pi^2}{2} \|f\|_0
\]

\[
\|D_2 S'f\|_0 \leq \frac{1}{2\pi} \|f\|_0 \left(\int_0^{2\pi} f(t, \xi) \, d\tau \, d\xi\right) = \frac{\pi^2}{2} \|f\|_0 \leq \frac{\pi^2}{2} \|f\|_0.
\]

327
From here follows that \( \|S'f\|_1 \leq (\pi^2/2) \|f\|_0 \) and \( \|S'f\|_{k+1} = \|S'f\|_1 \leq (\pi^2/2) \).

ad 2) From the form of the operator \( S \) it follows immediately that \( \|Sf\|_0 \leq (\pi^2/2) \). Differentiating the relation \( (2.13) \) we get

\[
D_1 S f(t, x) = \frac{1}{2} \int_x^\pi (f(t + x - s, s) - f(t - x + s, s)) \, ds ,
\]

\[
D_2 S f(t, x) = \frac{1}{2} \int_x^\pi (f(t + x - s, s) + f(t - x + s, s)) \, ds
\]

and from here \( \|D_i S f\|_0 \leq \pi \|f\|_0 \leq (\pi^2/2) \|f\|_0 (i = 1, 2) \).

We shall prove that for \( k \geq 1 \) and \( f \in \mathbb{N}_k \) there hold the relations:

\[
D_i^{k+1} S f = D_i S D_i^k f ,
\]

\[
D_i^k D_2 S f = D_2 S D_i^k f ,
\]

\[
D_i^{k+1} S f = \frac{1}{2} \sum_{i=0}^{k-1} (1 + (-1)^i) D_i^{k-1-i} D_i^i f + D_{b(k)} S D_i^k f ,
\]

where \( b(k) = 1 \) for odd \( k \) and \( b(k) = 2 \) for even \( k \).

The first and the second relations are obvious, the third one will be proved by induction with respect to \( k \).

For \( k = 1 \) we get by differentiating of the relation \( (2.17) \):

\[
D_1^2 S f = -f + D_1 S D_1 f .
\]

Let us suppose that the relation \( (2.20) \) holds for \( k = n \). Then

\[
D_i^{n+2} S f = D_2 (D_i^{n+1} S f) = -\frac{1}{2} \sum_{i=0}^{n-1} (1 + (-1)^i) D_i^{n-i} D_i^i f + D_2 D_{b(n)} S D_i^i f
\]

\( b(n) = 1 \) for odd \( n \) and then

\[
D_2 D_{b(n)} S D_i^i f = D_2 S D_i^{n+1} f = -\frac{1}{2} (1 + (-1)^n) D_2^o D_i^i f + D_{b(n+1)} S D_i^{n+1} f
\]

\( b(n) = 2 \) for even \( n \) and then

\[
D_2 D_{b(n)} S D_i^i f = D_2 S D_i^{n+1} f = -D_i^i f + D_1 S D_i^{n+1} f = -\frac{1}{2} (1 + (-1)^n) D_2^o D_i^i f + D_{b(n+1)} S D_i^{n+1} f .
\]

In both cases we get

\[
D_i^{n+2} S f = -\frac{1}{2} \sum_{i=0}^{n} (1 + (-1)^i) D_i^{n-i} D_i^i f + D_{b(n+1)} S D_i^{n+1} f .
\]

The inductive step is performed.
Let $f \in N_k$. We shall estimate $\|D^i_1 D^j_2 Sf\|_0$ for $i + j \leq k + 1$. If $j = 0$, $i \geq 1$, then it holds
\[
\|D^i_1 D^j_2 Sf\|_0 = \|D^i_1 D^j_2 D^{-1} f\|_0 \leq \pi \|D^i_1 D^{-1} f\|_0 \leq \frac{\pi^2}{2} \|f\|_k.
\]
If $j > 0$, then
\[
\|D^i_1 D^j_2 Sf\|_0 = \|D^i_1 D^j_2 D^{-1} f\|_0 = \| - \frac{1}{\pi} \sum_{m=0}^{j-2} (1 + (-1)^m) D^j_2 D^{-2-m} D^{i+1} f + D_{k(j-1)} D^{i+j-1} f\|_0 \leq \sum_{m=0}^{j-2} \|D^j_2 D^{-2-m} D^{i+1} f\|_0 + \pi \|D^{i+j-1} f\|_0 \leq (k + \pi^2/2) \|f\|_k
\]
and from here we get that for $k \geq 0$, $f \in N_k$ it holds
\[
\|Sf\|_{k+1} \leq \left( k + \frac{\pi^2}{2} \right) \|f\|_k.
\]
Lemma is proved.

Let us define the operator $A$ on $N$.

(2.21) $f \in N : Af = P_2(S + S')f$.

From the preceding it follows that for $k \geq 0$

(2.22) $A \in [N_k \rightarrow N_{k+1}]$, $\|Af\|_{k+1} \leq 3(k + \pi^2) \|f\|_k$.

Remark 2.1. $R(P_1) = R(Z)$ is by (2.1) a class of solutions of a homogeneous equation (2.2) and so the function $u = Af = (S + S')f - P_1(S + S')f$ for $f \in N_1$ is by (2.5), (2.4) a classical solution of the equation (2.2).

3. AUXILIARY THEOREMS

**Lemma 3.1.** Let $r_i$ be positive numbers ($i = 0, 1, \ldots, k + 1$, $k$ nonnegative integer) and let $M_1 = \{p \in C^k, |D^i p|_0 \leq r_i, i = 0, 1, \ldots, k\}$, $M_2 = \{p \in C^{k+1}, |D^i p|_0 \leq r_i, i = 0, 1, \ldots, k + 1\}$ (hence $M_2 \subset M_1$). Let $T$ be a continuous mapping of $M_1 \subset C^k$ into $C^k$, which maps $M_2$ into itself. Then there exists a fixed point of the operator $T$ in $M_1$.

**Proof.** The closure $\overline{M_2}$ of the set $M_2$ in the space $C^k$ is a convex and compact subset of $M_1 \subset C^k$ and by the assumptions of the lemma $T$ is a continuous mapping of $\overline{M_2}$ into itself. Hence, by the Schauder fixed point theorem $T$ has a fixed point in $\overline{M_2} \subset M_1$.

**Lemma 3.2.** Let the operator $I$ be given on $C^0$ by prescription

(3.1) $p \in C^0 : Ip(y) = \int_0^y p(s) \, ds + \int_0^{2\pi} \frac{s}{2\pi} p(s) \, ds, \quad y \in R$.  

329
Then it holds

\[(3.2) \quad 1) \quad D^{k+1} I p = D^k p, \quad (p \in C^k), \quad ID^{k+1} p = D^k p \quad (p \in C^{k+1}) \quad \text{for} \quad k \geq 0, \]

\[2) \quad I \in [C^k \rightarrow C^{k+1}], \quad |I p|_{k+1} \leq \frac{\pi}{2} |p|_k. \]

Proof. By an easy calculation we can verify that 1) holds and that \( I p \) is a unique primitive function of \( p \) for which \( [I p] = 0 \) and then \( I p \in C^{k+1} \) for \( p \in C^k \). To estimate the norm of \( I p \) let us remark that we can add to \( I p \) any function of the form \( \int_0^s f(y) \cdot p(s) ds = f(y) [p] = 0. \)

\[|I p(y)| = \left| \int_0^y p(s) ds + \int_0^{2\pi} \frac{s}{2\pi} p(s) ds - \int_0^{2\pi} \left( \frac{y}{2\pi} + \frac{1}{2} \right) p(s) ds \right| = \]

\[= \left| \int_0^y \left( \frac{s - y}{2\pi} + \frac{1}{2} \right) p(s) ds + \int_y^{2\pi} \left( \frac{s - y}{2\pi} - \frac{1}{2} \right) p(s) ds \right| \leq \]

\[\leq |p|_0 \left( \int_0^y \frac{s - y}{2\pi} + \frac{1}{2} ds + \int_y^{2\pi} \frac{s - y}{2\pi} \right) \frac{1}{2} ds \right) = \frac{\pi}{2} |p|_0. \]

From here and from 1) our assertion follows.

Lemma 3.3. Let \( p \in C^0, \quad J = (0, \pi), \quad g \) be a continuous and bounded function on \( J, \quad -\beta \leq g(s) \leq -\gamma < 0 \quad (s \in J). \) Let us denote \( J^+_k = \{ s \in J, \quad p(y) - p(y - 2ks) \geq 0 \}, \quad J^-_k = J \setminus J^+_k, \quad k \) a nonzero integer. Then it holds

\[\int_{J^+_k} p(y - 2ks) ds = -\int_{J^-_k} p(y - 2ks) ds, \quad \text{we obtain} \]

\[\int g(s) (p(y) - p(y - 2ks)) ds \leq \int_{J^+_k} g(s) (p(y) - p(y - 2ks)) ds + \]

\[+ \int_{J^-_k} g(s) (p(y - 2ks) - p(y)) ds + \pi \beta \sup_{s \in J^-_k} (p(y - 2ks) - p(y)) \leq -\gamma p(y) + \pi \beta \sup_{s \in J^+_k} (p(y - 2ks) - p(y)) \]

Proof. Because of \( \int_{J^+_k} p(y - 2ks) ds = -\int_{J^-_k} p(y - 2ks) ds, \) we obtain

\[\int g(s) (p(y) - p(y - 2ks)) ds \leq -\gamma \int_{J^+_k} p(y) ds + \]

\[+ \int_{J^-_k} g(s) (p(y - 2ks) - p(y)) ds + \pi \beta \sup_{s \in J^-_k} (p(y - 2ks) - p(y)) \leq -\gamma p(y) + \pi \beta \sup_{s \in J^+_k} (p(y - 2ks) - p(y)). \]
On the other hand

\[
\int_{J_k^+} g(s) (p(y) - p(y - 2ks)) \, ds + \int_{J_k^-} g(s) (p(y) - p(y - 2ks)) \, ds \geq
\]

\[
\geq -\beta \int_{J_k^+} (p(y) - p(y - 2ks)) \, ds - \gamma \int_{J_k^-} (p(y) - p(y - 2ks)) \, ds \geq
\]

\[
\geq -\pi \beta \sup_{s \in J_k^+} (p(y) - p(y - 2ks)) - \gamma p(y) m(J_k^-) - \gamma \int_{J_k^+} p(y) \, ds =
\]

\[
= -\pi \beta \sup_{s \in J_k^+} (p(y) - p(y - 2ks)) - \pi \gamma p(y).
\]

Lemma is proved.

**Lemma 3.4.** Let \( p \in C^0, J = (0, \pi), g \) be a continuous and bounded function on \( J, a \leq g(s) \leq b \) (\( s \in J \)), \( k \) a nonzero integer, \( J_k^+ = \{s \in J, p(y) + p(y - 2ks) \geq 0\}, J_k^- = J \setminus J_k^+ \). Then

\[
(3.4) \quad \pi \alpha p(y) + (b - a) p(y) m(J_k^-) - \frac{\pi}{2} (b - a) |p|_0 \leq
\]

\[
\leq \int_0^\pi g(s) (p(y) + p(y - 2ks)) \, ds \leq
\]

\[
\leq \pi \alpha p(y) + (b - a) p(y) m(J_k^+) + \frac{\pi}{2} (b - a) |p|_0.
\]

**Proof.**

\[
\int_J g(s) (p(y) + p(y - 2ks)) \, ds = \int_{J_k^+} + \int_{J_k^-} \leq b \int_{J_k^+} p(y) m(J_k^+) + b \int_{J_k^-} p(y - 2ks) \, ds +
\]

\[
+ a \int_{J_k^-} p(y) m(J_k^-) + a \int_{J_k^-} p(y - 2ks) \, ds = p(y) (bm(J_k^+) + a m(J_k^-)) +
\]

\[
+ (b - a) \int_{J_k^+} p(y - 2ks) \, ds \leq \pi \alpha p(y) + (b - a) p(y) m(J_k^+) + \frac{\pi}{2} (b - a) |p|_0.
\]

On the other hand

\[
\int_{J_k^+} \pi(s) (p(y) + p(y - 2ks)) \, ds + \int_{J_k^-} g(s) (p(y) + p(y - 2ks)) \, ds \geq
\]
\[ a p(y) m(J^+_k) + a \int_{J^+_k} p(y - 2ks) \, ds + b p(y) m(J^-_k) + b \int_{J^-_k} p(y - 2ks) \, ds = \]
\[ = p(y) (a m(J^+_k) + b m(J^-_k)) + (b - a) \int_{J^-_k} p(y - 2ks) \, ds \geq \]
\[ \geq \pi a p(y) + (b - a) p(y) m(J^-_k) - \frac{\pi}{2} (b - a) |p|_0. \]

Let us conclude this paragraph by some estimates of the norm of a composite function.

Let \( f \) be a function of \((n + 2)\) real variables and let \( u_m \in C_k \) \((m = 1, \ldots, n)\). Let us denote \( f[u_1, \ldots, u_n] \) the function defined on \( G \) by

\[(3.5) \quad f[u_1, \ldots, u_n](t, x) = f(t, x, u_1(t, x), \ldots, u_n(t, x)) . \]

Then it holds: Each derivative of the function \( f[u_1, \ldots, u_n] \) of the order \( l \leq k \) has not more than \( l! (n + 2)^l \) members each of them being estimated at the point \((t, x)\) by

\[ \sup_{0 \leq |i| \leq l} \left| D^i f(t, x, u_1(t, x), \ldots, u_n(t, x)) \right| \left( \max \left( \sup_{(t, x) \in G} |D^i u_1(t, x)|, 1 \right) \right)^l \]

where \( i \) denotes the vector \((i_1, i_2, \ldots, i_{n+2})\), \( i_m \) a nonnegative integers, \(|i| = \sum_{m=1}^{n+2} i_m \)

and \( D^i \) denotes the derivative \( D_1^{i_1} D_2^{i_2} \ldots D_{n+2}^{i_{n+2}} \).

If \( f \) is such that for any \( \varrho > 0 \)

\[(3.6) \quad F_f(k, \varrho) = \sup_{|i| \leq k} \sup_{m \leq \varrho} \left| D^i f(t, x, \alpha_1, \ldots, \alpha_n) \right| < +\infty \]

then for any \( u_m \in C_k, \|u_m\|_k \leq r, \|u_m\|_0 \leq r_0 \) \((m = 1, \ldots, n, r \geq 1)\) the function \( f[u_1, \ldots, u_n] \) belongs to \( C_k \) and

\[(3.7) \quad \|f[u_1, \ldots, u_n]\|_k \leq k! (n + 2)^k F_f(k, r_0) r^k. \]

Let us denote \( K_f(k, r_0, r) = k! (n + 2)^k F_f(k, r_0) r^k \).

Let \( u_m, v_m \in C_k, \|u_m\|_k \leq r, \|v_m\|_k \leq r, \|u_m\|_0 \leq r_0, \|v_m\|_0 \leq r_0 \) \((m = 1, \ldots, n)\) and let \( D^i f \) be continuous for \(|i| \leq k + 1\). Then from the mean-value theorem we obtain

\[ f[u_1, \ldots, u_n] - f[v_1, \ldots, v_n] = \sum_{m=1}^{n} g_m(u_m - v_m) \]

where \( g_m(t, x) = \int_0^1 D_{m+1}^{i_{m+1}} f[v_1, \ldots, v_{m-1}, v_m + \varrho(u_m - v_m), u_{m+1}, \ldots, u_n](t, x) \, dq \)

Evidently the functions \( g_m \in C_k \) and \( g_m \leq K_f(k + 1, r_0, r) \). For \( i + j \leq k \) we
Thus the following lemma holds:

**Lemma 3.5.** Let $D_i f$ be continuous for $|i| \leq k + 1$ and let the condition (3.5) be fulfilled. Then for $u_m, v_m \in C_k$, $\|u_m\|_0 \leq r_0$, $\|v_m\|_0 \leq r_0$, $\|u_m\|_k \leq r$, $\|v_m\|_k \leq r$ $(m = 1, \ldots, n)$ the following estimates hold

\[(3.8)\]
1) $\|f[u_1, \ldots, u_n]\|_k \leq K_f(k, r_0, r)$,

\[(3.9)\]
2) $\|f[u_1, \ldots, u_n] - f[v_1, \ldots, v_n]\|_k \leq 2^k K_f(k + 1, r_0, r) \sum_{m=1}^{n} \|u_m - v_m\|_k$,

where $K_f(k, r_0, r) = k! (n + 2)^k F_f(k, r_0) r^k$, $F_f(k, r_0)$ is given by (3.6).

### 4. NONLINEAR EQUATION

Let us solve the problem (1.1), (1.2) under the assumptions:

1° $D_i f$ are continuous for $|i| \leq k + 1$ and the assumption (3.6) is fulfilled.

2° There exist $\gamma > 0$, $r_0 > 0$ such that

\[(4.1)\]
a) $D_4 f \leq -\gamma < 0$ on $G_2 = G \times \langle -\pi r_0, \pi r_0 \rangle \times \langle -2r_0, 2r_0 \rangle \times \langle -2r_0, 2r_0 \rangle$,

\[(4.2)\]
\[d = \gamma r_0 - \sup_{G_2} \{|f(t, x, u, 0, w)|, (t, x) \in G, |u| \leq \pi r_0, |w| \leq 2r_0 > 0,\]

\[(4.3)\]
c) $\sup_{G_2} D_5 f - \inf_{G_2} D_5 f = -\gamma < 0$.

Let $f[p, u]$ denote the function

\[(4.4)\]
\[f[p, u] = f[Z_1 p + u, Zp + D_1 u, Z_1 p + D_2 u],\]

where $Z_1$ is the operator defined by $Z_1 p(t, x) = p(t + x) + p(t - x)$.

We shall prove the existence of a $2\pi$-periodic solution of the problem (1.1), (1.2) in the following way: First we shall prove that if $\varepsilon$ is sufficiently small and $p \in C^{k-1}$ then there exists a function $a'(p) \in C_k$ which satisfies the equation $(D_1^2 - D_2^2)(Z_1 p + a'(p)) = \varepsilon P_2 f[p, a'(p)]$ and further we shall seek such $p$ for which $P_2 f[p, a'(p)] = f[p, a'(p)]$. 

333
Let \( r_i \) \((i = 0, \ldots, k)\) be positive numbers, \( r = \pi \max (r_i, 0 \leq i \leq k) + 1. \) Let us denote for \( i = 0, \ldots, k \)

\[
A_i = \{ p \in C^1, |D^j p|_0 \leq r_j, j = 0, \ldots, i\}, \quad B_i = \{ u \in N_i, \|u\|_{i+1} \leq 1\}
\]

(then \( A_{i+1} \subset A_i, B_{i+1} \subset B_i \)).

**Lemma 4.1.** The equation

\[
u = \varepsilon A P_2 f(p, u)
\]

has for \( p \in A_0 \) and \( \varepsilon < [54 2^h(k + \pi^2) K_f(k + 1, \pi r_0 + 1, r)]^{-1} \) a unique solution \( a^\varepsilon(p) \in B_0 \) and further it holds

1) \( a^\varepsilon(p) \in B_i \) for \( p \in A_i \) \((i = 0, \ldots, k)\),

2) \( \|a^\varepsilon(p)\|_{i+1} \leq \varepsilon K_1 \) \((p \in A_i)\),

3) \( \|a^\varepsilon(p) - a^\varepsilon(q)\|_{i+1} \leq \varepsilon K_2 |p - q|_i \) \((p, q \in A_i)\),

where \( K_1 = 9(k + \pi^2) K_f(k, \pi r_0 + 1, r) \), \( K_2 = 18(k + \pi^2) (\pi + 4) 2^h K_f(k + 1, \pi r_0 + 1, r) \).

**Proof.** Let \( p \in A_i \) \((0 \leq i \leq k)\). Then by (2.11) and (2.22) \( \varepsilon A P_2 f\{p, u\} \) maps \( N_{i+1} \) into itself. Using lemma 3.4 we get for \( u \in B_i \)

\[
\|\varepsilon A P_2 f\{p, u\}\|_{i+1} \leq \varepsilon (i + \pi^2) 3\|f\{p, u\}\|_i \leq \varepsilon 9(k + \pi^2) K_f(k, \pi r_0 + 1, r) \leq 1
\]

and so \( \varepsilon A P_2 f\{p, u\} \) maps \( B_i \) into itself. Further for \( u \in B_i, v \in B_i \)

\[
\|\varepsilon A P_2 f\{p, u\} - \varepsilon A P_2 f\{p, v\}\|_{i+1} \leq \varepsilon 9(i + \pi^2) \|f\{p, u\} - f\{p, v\}\|_i \leq \\
\leq \varepsilon 9(k + \pi^2) 2^h K_f(k + 1, \pi r_0 + 1, r) \|u - v\|_i + \|D_1 u - D_1 v\|_i + \\
+ \|D_2 u - D_2 v\|_i \leq \varepsilon 27(k + \pi^2) 2^h K_f(k + 1, \pi r_0 + 1, r) \|u - v\|_{i+1} \leq \\
\leq \frac{1}{2} \|u - v\|_{i+1}.
\]

We get that \( \varepsilon A P_2 f\{p, u\} \) is a contraction on \( B_0 \) for \( p \in A_0 \). Hence there exists a unique solution \( a^\varepsilon(p) \in B_0 \) of the equation (4.6). As for \( p \in A_i \) the operator \( \varepsilon A P_2 f\{p, u\} \) is also a contraction in \( B_i \), there exists for \( p \in A_i \subset A_0 \) a solution of the equation (4.6) in \( B_i \subset B_0 \) and from the uniqueness in \( B_0 \) it follows that \( a^\varepsilon(p) \in B_i \) for \( p \in A_i \).

The assertion 2) follows immediately from (4.9).
The solution \( a'(p) \) we can get by the method of successive approximations: \( u_0 = a'(q) \), \( u_{n+1} = eAP \{ f \{ p, u_n \} \} \). By the well known estimates, using (4.10), we get

\[
\|a'(p) - a'(q)\|_{i+1} = \lim_{n \to \infty} \|u_n - u_0\|_{i+1} \leq 2\|u_1 - u_0\|_{i+1} \leq 2\|eAP \{ f \{ p, a'(q) \} \} - f \{ q, a'(q) \}\|_i \leq \varepsilon 18(i + \pi^2) 2^k K_j(k + 1, \pi r_0 + 1, r)(\|ZI(p - q)\|_i + \|Z(p - q)\|_i + \|Z(p - q)\|_i) \leq \varepsilon 18(k + \pi^2)(4 + \pi) 2^k K_j(k + 1, \pi r_0 + 1, r) |p - q|_i.
\]

Lemma is proved.

Now let us solve the equation

(4.11) \[ P_2 f \{ p, a'(p) \} = f \{ p, a'(p) \}, \]

where \( a'(p) \) is defined in lemma 4.1.

We shall solve this equation with help of lemma 3.1. The role of the sets \( M_1, M_2 \) respectively will play the sets \( \mathcal{A}_1, \mathcal{A}_2 \) respectively with \( r_0 \) fulfilling the assumption \( 2^a \) and \( r_1 \) which are given by the recurrent formula

(4.12) \[ r_{i+1} = \frac{1}{\alpha} (F_f(1, \pi r_0) + (i + 1)! 8^{i+1} 2 F_f(i + 1, \pi r_0). \]

Further let \( r = \pi \max r_i + 1, A_i \) and \( \varepsilon \) be as in lemma 4.1 and besides it \( \varepsilon < < \min (\alpha, d) (2^k K_j(k + 1, \pi r_0 + 1, r) K_1)^{-1} \), where \( K_1 \) is given by (4.7).

The equation (4.11) is by 5) of lemma 2.1 equivalent to the equation

(4.13) \[ Qf \{ p, a'(p) \} = 0. \]

Let \( T_1, T_2 \) denote the operators defined on \( \mathcal{A}_0 \)

(4.14) \[ T_1 p = Qf \{ p, 0 \}, \]

(4.15) \[ T_2 p = Q(f \{ p, a'(p) \} - f \{ p, 0 \}) \]

and for \( \delta > 0 \)

(4.16) \[ T_\delta p = (E + \delta T_1 + \delta T_2) p. \]

According to lemma 2.1 and lemma 4.1 the mappings \( T_1, T_2 \) map \( \mathcal{A}_i \) into \( C^l \). Further it is obvious that to solve the equation (4.13) means to find a fixed point of the operator \( T_\delta \) for some \( \delta > 0. \)
Using (2.9), (3.9), (4.7) we get the estimate for the operator $T_2$

\begin{equation}
(4.17) \quad |T_2 p| \leq |Q(f(p, a^2(p)) - f(p, 0))| \leq \|f(p, a^2(p)) - f(p, 0)\| \leq 2^{i+1} K_f(i + 1, \pi r_0 + 1, r) \|a^2(p)\|_{i+1} \leq \varepsilon K_3,
\end{equation}

where $K_3 = 2^{i+1} K_f(k + 1, \pi r_0 + 1, r) K_1$ ($K_1$ is given by (4.7)).

Let $\eta = \min \left(\left[d - \varepsilon K_3\right] F_f(1, \pi r_0), 0 \right]$, $0 \leq \delta \leq \delta_0 \leq \min (\eta F_f(1, \pi r_0), + \varepsilon K_3)^{-1}, \gamma^{-1})$ and let us prove that $T_3$ maps $A_0$ into itself. If $y$ is such that $\|p(y)\| \leq r_0 - \eta$, then

\begin{equation}
|T_3 p(y)| = \left| p(y) + \frac{\delta}{2\pi} \int_0^\pi \left[ f(p, 0) (y - s, s) - f(p, 0) (y + s, s) \right] ds + \delta T_2 p(y) \right| \leq r_0 - \eta + \delta F_f(0, \pi r_0) + \delta \varepsilon K_3 \leq r_0.
\end{equation}

If $r_0 - \eta \leq p(y) \leq r_0$, then from the same expression we obtain the estimate: $T_3 p(y) \leq r_0 - \eta - \delta F_f(0, \pi r_0) - \delta \varepsilon K_3 \leq -r_0$. Using the mean-value theorem we get the operator $T_3$ in the form

\begin{equation}
T_3 p(y) = p(y) + \frac{\delta}{2\pi} \int_0^\pi \left[ g_1(y, s) (p(y) - p(y - 2s)) + g_2(y, s) (p(y) - p(y + 2s)) \right] ds +
\end{equation}

\begin{equation}
+ \frac{\delta}{2\pi} \int_0^\pi \left( f[ZIp, 0, Z_1p] (y - s, s) - f[ZIp, 0, Z_1p] (y + s, s) \right) ds + \delta T_2 p(y),
\end{equation}

where $g_m(y, s) = \int_0^s D_4 f[ZIp, qZp, Z_1p] (y + (-1)^m s, s) dq$ ($m = 1, 2$).

From this expression we get for $r_0 - \eta \leq p(y) \leq r_0$ by lemma 3.3 the estimate

\begin{equation}
T_3 p(y) \leq p(y) - \delta \gamma p(y) + \delta F_f(1, \pi r_0) \eta +
\end{equation}

\begin{equation}
+ \delta \sup \{|f(t, x, u, 0, w)|, (t, x) \in G, |u| \leq \pi r_0, |w| \leq 2r_0\} +
\end{equation}

\begin{equation}
+ \delta \varepsilon K_3 \leq r_0 + \delta(-d + \eta F_f(1, \pi r_0) + \varepsilon K_3) \leq r_0.
\end{equation}

In a similar way (using the first inequality in lemma 3.3) we get for $-r_0 \leq p(y) \leq r_0$ that $-r_0 \leq T_3 p(y) \leq r_0$ and thus $T_3 p \in A_0$ for $p \in A_0$.

Let us assume that there exist such $\delta_j > 0$, $0 \leq j \leq i \leq k - 1$, that for $0 < \delta \leq \delta_j$ the operator $T_3$ maps the set $A_j$ into itself and let us seek $\delta_{i+1}$ such that $T_3$ for $0 < \delta \leq \delta_{i+1}$ maps $A_{i+1}$ into itself.

\begin{equation}
(4.18) \quad D^{i+1} T_3 p(y) = D^{i+1} p(y) +
\end{equation}

\begin{equation}
+ \frac{\delta}{2\pi} \int_0^\pi \left[ D_4 f[p, 0] (y - s, s) (D^{i+1} p(y) - D^{i+1} p(y - 2s)) +
\end{equation}

\begin{equation}
+ D_4 f[p, 0] (y + s, s) (D^{i+1} p(y) - D^{i+1} p(y + 2s)) \right] ds
\end{equation}
$$+ D_s f(p, 0) (y - s, s) (D^{i+1} p(y) + D^{i+1} p(y - 2s)) -$$

$$- D_s f(p, 0) (y + s, s) (D^{i+1} p(y) + D^{i+1} p(y + 2s))] ds +$$

$$+ \delta X_{i+1} (y) + \delta D^{i+1} T_2 p(y),$$

where $X_{i+1}$ is the sum of at most $(i + 1)! 4^{i+1}$ members of the form

$$\frac{1}{2\pi} \int_0^s \left[ D^nf(p, 0) (y - s, s) h(y - s, s) - D^nf(p, 0) (y + s, s) h(y + s, s) \right] ds,$$

$|n| \leq i + 1$ and $h$ is the product of at most $i + 1$ members $D^j_1 Z^j p, D^j_1 Z^j p, D^j_1 Z^j p$

$(1 \leq j \leq i + 1, 1 \leq k \leq i, 1 \leq l \leq i)$ and from here an estimate for $X_{i+1}$ follows

$$(4.19) \quad |X_{i+1}| \leq (i + 1)! 8^{i+1} 2F_i (i + 1, \pi r_0) \left[ \max (r_0, \ldots, r_i) \right]^{i+1} \equiv c_{i+1}.$$

Let us suppose that $p \in A_{i+1}$. Then for $y$ for which $|D^{i+1} p(y)| \leq r_{i+1} - 1$ we get by (4.18), (4.19)

$$(4.20) \quad D^{i+1} T_3 p(y) \leq r_{i+1} - 1 + \delta (2F_i (1, \pi r_0) 2r_{i+1} + c_{i+1} + \varepsilon K_3) \leq r_{i+1}$$

if $0 < \delta < \delta_{i+1} = \min \left[ \delta_0, (4F_i (1, \pi r_0) r_{i+1} + c_{i+1} + \varepsilon K_3)^{-1} \right]$.

If $r_{i+1} - 1 \leq D^{i+1} p(y) \leq r_{i+1}$, then from (4.18) we get $D^{i+1} T_3 p(y) \geq -r_{i+1}$

and further by lemma 3.3 and lemma 3.4

$$(4.8) \quad D^{i+1} T_3 p(y) \leq r_{i+1} + \delta (-\gamma r_{i+1} + F_i (1, \pi r_0)) +$$

$$+ \left( \sup_{G_1} D_s f(p, 0) - \inf_{G_2} D_s f(p, 0) \right) r_{i+1} + c_{i+1} + \varepsilon K_3 \leq$$

$$\leq r_{i+1} + \delta (-\gamma r_{i+1} + F_i (1, \pi r_0) + c_{i+1} + \varepsilon K_3) \leq r_{i+1}.$$

For $-r_{i+1} \leq D^{i+1} p(y) \leq -r_{i+1} + 1$ we proceed analogously and finally we get

$$|D^{i+1} T_3 p(y)| \leq r_{i+1} \text{ if } |D^{i+1} p(y)| \leq r_{i+1}.$$

Thus we have proved that for $\delta$ fulfilling (4.20) and $r_{i+1}$ given by (4.12) the operator $T_3$ maps the set $A_{i+1}$ into itself.

$T_3$ is a continuous mapping on $C^k$ and from above it follows that it fulfills the assumptions of lemma 3.1 with $M_1 = A_{k-1}$ and $M_2 = A_k$. Thus there exists a fixed-point $p_0 \in A_{k-1}$ of the operator $T_3$. This $p_0$ satisfies the equation (4.13) and hence

$$a^\varepsilon(p_0) = \varepsilon Af \{ p_0, a^\varepsilon(p_0) \}.$$

From remark 2.1 it follows that the function $u_\varepsilon = Z I p_0 + a^\varepsilon(p_0)$ is for $k \geq 2$

a classical solution of the problem (1.1), (1.2). We have proved the following theorem

**Theorem 1.** Let $f$ be defined on $G_1 = R \times (0, \pi) \times R \times R \times R$ and fulfill the following assumptions:

1) $f$ has derivatives up to the order $k + 1$ and for $r > 0$ sup sup $\sup_{|I| \leq k+1} |D^I f(t, x, u, v, w)|,

$$(t, x) \in G, |u| \leq r, |v| \leq r, |w| \leq r < +\infty.$$
2) There exist $r_0 > 0$ and $\gamma > 0$ such that

a) $D_4f \leq -\gamma < 0$ on $G_2 = G \times \langle -\pi r_0, \pi r_0 \rangle \times \langle -2r_0, 2r_0 \rangle \times \langle -2r_0, 2r_0 \rangle$,

b) $\sup_{G_2} \frac{D_4f}{r_0} - \inf_{G_2} \frac{D_4f}{r_0} = -\alpha < 0$,

c) $d = \gamma r_0 - \sup \{|f(t, x, u, 0, w)|, (t, x) \in G, |u| \leq \pi r_0, |w| \leq 2r_0\} > 0$.

Then there exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$ there exists the function $u_\varepsilon \in C_k$ which is a solution of the problem (1.1), (1.2).

5. ANOTHER NONLINEAR CASE

We shall solve the equation

\begin{equation}
(5.1) \quad u_{tt} - u_{xx} = \varepsilon f(t, x, u, \varepsilon)
\end{equation}

with the boundary conditions (1.2). Let $f$ be defined on $G_3 = R \times (0, \pi) \times R \times R \times \langle 0, \varepsilon_0 \rangle$ and fulfill the following assumptions

1) $D_3^j D_3^k D_3^l f$, $i_1 + i_2 + i_3 \leq k + 1$, are defined and continuous on $G_3$.

2) There exists $\gamma > 0$ such that

\begin{equation}
(5.2) \quad D_3f \leq -\gamma < 0 \quad \text{on} \quad G_4 = G \times \langle -2r_0 - 1, 2r_0 + 1 \rangle \times \langle 0, \varepsilon_0 \rangle,
\end{equation}

where

\begin{equation}
(5.3) \quad \frac{r_0}{\gamma} > \frac{1}{\gamma} \sup \langle |f(t, x, 0, 0)|; (t, x) \in G \rangle.
\end{equation}

In this paragraph $f[u, \varepsilon](t, x) = f(t, x, u(t, x), \varepsilon)$.

As in the preceding case we shall seek a solution $a^t(p)$ of the equation $u = \varepsilon AP_2 f[Zp + u, \varepsilon]$ and then we shall prove that there exists a function $p$ such that $P_2 f[Zp + a^t(p), \varepsilon] = f[Zp + a^t(p), \varepsilon]$ with help of the implicit function theorem. We could proceed in the same way as in paragraph 4, but using the implicit function theorem we obtain immediately a continuous dependence of the solution on $\varepsilon$.

Similarly as in lemma 4.1 for $\varepsilon < \bar{\varepsilon} = \min \bigl(\frac{18(k + \pi^2) 2^{k+1} K_f(k, 1, 2r_0 + 1, r)}{(k + 1, 2r_0 + 1, r)}, \rho_0 \bigr)$ and $p \in A_{i+1} = \{p \in C^i, |D^j p| \leq \rho_j, j = 0, \ldots, i\}$ the operator $\varepsilon AP_2 f[Zp + u, \varepsilon]$ is a contraction on $B_i = \{u \in N_{i+1}, |u|_{i+1} \leq 1\}$, hence we get a unique solution $a^t(p) \in B_i$ for $p \in A_{i+1}$ for which

\begin{equation}
(5.4) \quad 1) \quad \|a^t(p)\|_{i+1} \leq \varepsilon K_1,
\end{equation}

\begin{equation}
(5.5) \quad 2) \quad \|a^t(p) - a^t(q)\|_{i+1} \leq \varepsilon K_2 |p - q|,\n\end{equation}

where $K_1 = 9(k + \pi^2) K_f(k, 2r_0 + 1, r)$, $K_2 = 18(k + \pi^2) 2^{k+1} K_f(k + 1, 2r_0 + 1, r)$, $r = 2 \max_{0 \leq i \leq k} r_i + 1$. 

338
Further we shall prove that $a'(p)$ is continuous for $(p, \varepsilon) \in A \times \langle 0, \bar{\varepsilon} \rangle$ and that there exists G-derivative $a''(p)$ continuous in $(p, \varepsilon)$. As (5.5) holds, it suffices to prove that for a fixed $p$ the function $a'(p)$ is continuous in $\varepsilon$. Let $\varepsilon_1, \varepsilon_2 \in \langle 0, \bar{\varepsilon} \rangle$. Then $a''(p)$ we get from $a''(p)$ by the method of successive approximations: $u_0 = a''(p)$, $u_{n+1} = e_1 A P_2 f[Zp + u_n, \varepsilon_1]$. Then

$$
\|a''(p) - a''(p)\|_{k+1} = \lim_{n \to \infty} \|u_n - u_0\|_{k+1} \leq 2\|u_1 - u_0\|_{k+1} \leq 18(k + \pi^2) \|\varepsilon_1 f[Zp + a''(p), \varepsilon_1] - \varepsilon_2 f[Zp + a''(p), \varepsilon_2]\|_{k} \leq 18(k + \pi^2) [(\varepsilon_1 - \varepsilon_2) \|f[Zp + a''(p), \varepsilon_1]\|_{k} + + \varepsilon_2 \|f[Zp + a''(p), \varepsilon_1]\|_{k} - f[Zp + a''(p), \varepsilon_2]\|_{k} \leq \omega(\varepsilon_1 - \varepsilon_2),
$$

where $\omega$ is a function on $\langle 0, \bar{\varepsilon} \rangle$, continuous in 0 and $\omega(0) = 0, \omega$ depends on $f, k$ and $r$.

To prove the existence of $a''(p)$ let us note that the function $v'(p) = a'(p) + Zp$ satisfies the equation

$$
v'(p) = Zp + \varepsilon A P_2 f[v'(p), \varepsilon].
$$

Then according to the known theorem (see e.g. [4]) there exists for $\varepsilon$ sufficiently small ($\varepsilon < (\|A\| \|P_2\| \sup_{G} D_3 f)^{-1}$) a G-derivative $v''(p) = [E - \varepsilon R_e(p)]^{-1} Zq$ and hence $a''(p)$ has a G-derivative

$$
a''(p)(q) = ([E - \varepsilon R_e(p)]^{-1} - E) Zq,
$$

where

$$
R_e(p)(w) = A P_2 (D_3 f[v''(p), \varepsilon] w).
$$

It is obvious that this derivative is continuous in $p$ and $\varepsilon$.

Let $\bar{\varepsilon} \leq \bar{\varepsilon}$ be such that all above assumptions are fulfilled for $\varepsilon < \bar{\varepsilon}$. Let us denote

$$
V(p, \varepsilon) = Q f[Zp + a'(p), \varepsilon]
$$

and let us prove that the operator $V$ fulfils the assumptions of the implicit function theorem.

The operator $V$ maps $C^k \times \langle 0, \bar{\varepsilon} \rangle$ into $C^k$. By lemma 3.1 we shall prove that the equation $V(p, 0) = 0$ has a unique solution $p_0 \in C^k$. As in the preceding paragraph we shall prove the existence of a fixed point of the operator $T_\varepsilon p = p + \delta V(p, 0)$. Let $c = \sup \{|f(t, x, 0, 0)| (t, x) \in G\}$ and $r_0 > c/\gamma$. Further let $r_i (i = 1, \ldots, k + 1)$ be given by recurrent formulas

$$
r_i = \max \left(\frac{1}{\gamma} F_f(1, 2r_0) + \frac{1}{\gamma} 2^i(i + 1)! F_f(1, 2r_0) \left[\max\{r_0, \ldots, r_{i-1}\}\right]^i, 1\right).
$$

Let us denote $M_1 = \{p \in C^k, |D^i p|_0 \leq r_i, i = 0, \ldots, k\}, M_2 = \{p \in C^{k+1}, |D^i p|_0 \leq r_i, i = 0, \ldots, k + 1\}$. Let $p \in M_2$. We shall prove that also $T_\varepsilon p \in M_2$. 

339
If \( 0 < \eta < r_0 \), then for \( y \) such that \( |p(y)| \leq r_0 - \eta \) we get \( |T_\delta p(y)| \leq r_0 - \eta + \delta F_f(0,2r_0) \).

From the mean-value theorem we get the operator \( T_\delta \) in the form

\[
T_\delta p(y) = p(y) + \frac{\delta}{2\pi} \int_0^\pi \left[ g_1(y,s) (p(y) - p(y - 2s)) + g_2(y,s) (p(y) - p(y + 2s)) \right] ds + \frac{\delta}{2\pi} \int_0^\pi \left[ f(y - s, s, 0, 0) - f(y + s, s, 0, 0) \right] ds,
\]

where

\[
g_m(y,s) = \int_0^1 D_f[zp,0] \left( y + (-1)^m s, s \right) dq.
\]

Then by lemma 3.3 we get for \( r_0 - \eta \leq p(y) \leq r_0 \)

\[
T_\delta p(y) \leq p(y) - \delta y p(y) + \delta F_f(1,2r_0) + \delta c.
\]

Obviously for such \( y \)

\[
T_\delta p(y) \geq r_0 - \eta - F_f(0,2r_0).
\]

If \( 0 < \eta < \min \left[ (y r_0 - c) (F_f(1,2r_0))^{-1}, r_0 \right], 0 < \delta \leq \delta_0 = \min \left[ \eta (F_f(0,2r_0))^{-1}, \gamma^{-1} \right] \), then \( |T_\delta p(y)| \leq r_0 \). In the same way we can make the estimates if \( -r_0 \leq \eta \leq p(y) \leq -r_0 + \eta \) and hence \( |T_\delta p|_0 \leq r_0 \).

For \( i \geq 1 \) we have

\[
D^i T_\delta p(y) = D^i p(y) + \frac{\delta}{2\pi} \int_0^\pi \left[ D^i f[zp,0] \left( y - s, s \right) (D^i p(y) - D^i p(y - 2s)) \right] ds + \delta X_i(y),
\]

where \( |X_i(y)| \) is estimated by \( c_i = 2^i (i + 2) ! F_f(i, 2r_0) \left[ \max (r_0, \ldots, r_{i-1}) \right]^i \).

Now if we choose \( \eta = 1 \) and \( 0 < \delta \leq \delta_1 = \min \left( \delta_{i-1}, (2F_f(1,2r_0) r_i + c_i)^{-1} \right) \) we can prove similarly as above that \( |D^i T_\delta p|_0 \leq r_i \). Then the mapping \( T_\delta \) fulfils the assumptions of lemma 3.1 and hence there exists a fixed point \( p_0 \in M_1 \) of the operator \( T_\delta \) which is a solution of the equation \( V(p,0) = 0 \).

This \( p_0 \) is unique in \( C^0 \), because if \( p_1 \in C^0 \) is another solution of the equation \( V(p,0) = 0 \), then for \( p' = p_0 - p_1 \) it holds

\[
0 = V(p_0,0) - V(p_1,0) = \frac{1}{2\pi} \int_0^\pi \left[ g_1(y,s) \left( p'(y) - p'(y - 2s) \right) + g_2(y,s) \left( p'(y) - p(y + 2s) \right) \right] ds.
\]
where
\[ g_m(y, s) = \int_0^1 D_3f[Zp_0 + g(Zp_1 - Zp_0)] (y + (-1)^m s, s) \, dq. \]

From this expression we get for \( p' \neq \text{const} \) and \( y_0 \) such that \( p'(y_0) = \max (p'(y); y \in R) \), \( V(p_0, 0) (y_0) - V(p_1, 0) (y_0) < 0 \) which is a contradiction and hence \( p' = \text{const} = 0 \) because \( [p'] = 0 \).

From above it follows that the operator \( V(p, \varepsilon) \) is continuous in \( p \) and \( \varepsilon \) in a neighbourhood of \( (p_0, 0) \) and that it has a G-derivative \( V_p(p, \varepsilon) \) continuous in \( p \) and \( \varepsilon \) in the neighbourhood of \( (p_0, 0) \). Further we must prove that the operator \( H = V_p(p_0, 0) \) has an inverse operator \( H^{-1} \). It is easily seen that \( H \) maps \( C^k \) into itself. We shall prove that \( H \) is an \( 1-1 \) mapping. Let \( p \in C^k \) be such that \( Hp = 0 \). Let \( p(y_0) = \max (p(y); y \in R) \). If for some \( s p(y_0 - 2s) < p(y_0) \) or \( p(y_0 + 2s) < p(y_0) \), then
\[
0 = \int_0^\pi [D_3f[Zp_0, 0] (y - s, s) (p(y_0) - p(y_0 - 2s)) +
+ D_3f[Zp_0, 0] (y + s, s) (p(y_0) - p(y_0 + 2s))] \, ds < 0 .
\]

This is a contradiction and hence \( p = \text{const} = 0 \) because \( [p] = 0 \).

Let us denote \( g(y, s) = D_3f[Zp_0, 0] (y - s, s) + D_3f[Zp_0, 0] (y + s, s), g_0(y) = \int_0^\pi g(y, s) \, ds \). Then we can write the operator \( H \) as a sum \( H = H_1 + H_2 \), where
\[
H_1 \ p(y) \ = \frac{1}{2\pi} \int_0^{2\pi} g_0(s) p(s) \, ds ,
\]
\[
H_2 \ p(y) \ = \frac{1}{2\pi} \int_0^{2\pi} g_0(s) p(s) \, ds - \frac{1}{2\pi} \int_0^\pi (D_3f[Zp_0, 0] (y - s, s) p(y - 2s) +
+ D_3f[Zp_0, 0] (y + s, s) p(y + 2s)) \, ds .
\]

Evidently \( H_1, H_2 \) are the operators from \( C^k \) into \( C^k \), \( H_1 \) has on \( C^k \) a bounded \( H_1^{-1} \)
\[
H_1^{-1} \ p(y) \ = \frac{1}{g_0(y)} \left( p(y) - \left( \int_0^{2\pi} \frac{1}{g_0(s)} \, ds \right)^{-1} \int_0^{2\pi} \frac{1}{g_0(s)} p(s) \, ds \right) .
\]

We shall prove that the operator \( H_2 \) is completely continuous. Let \( U \) be a bounded set of \( C^k \). To prove that \( H_2(U) \) is compact in \( C^k \) it suffices to prove that the derivatives of the order \( k \) of functions from \( H_2(U) \) fulfil the assumptions of Arzela’s theorem.

It is obvious that they are uniformly bounded. Further (if \( k \geq 1 \))
\[
D^kH_2 \ p(y) \ = \ - \frac{1}{2\pi} \int_0^{2\pi} [D_3f[Zp_0, 0] (y - s, s) D^k p(y - 2s) +
+ D_3f[Zp_0, 0] (y + s, s) D^k p(y + 2s)] \, ds + X_k =
\]

341
In $X_k$ are only derivatives of $p$ up to the order $k - 1$, so they are equicontinuous. Further it is easily seen that the first and second integrals are also equicontinuous with respect to $p \in U$. So the operator $H_2$ is completely continuous. As the operator $E + H_1^{-1}H_2 = H_1^{-1}H$ is also an $1 - 1$ operator and $H_1^{-1}H_2$ is a completely continuous operator, there exists by the well known theorem a linear bounded $(E + H_1^{-1}H_2)^{-1}$ on $C^k$ and then there exists on $C^k$ also the linear bounded $H^{-1} = (E + H_1^{-1}H_2)^{-1}H_1^{-1}$.

Now we have verified all assumptions of the implicit function theorem and hence the following theorem holds:

**Theorem 2.** Let $f$ be defined on $G_3 = R \times (0, \pi) \times R \times \langle 0, \epsilon_0 \rangle$ and fulfill the following assumptions:

1) $D_1^{i_1}D_2^{i_2}D_3^{i_3}f$, $i_1 + i_2 + i_3 \leq k + 1$, are defined and continuous on $G_3$.

2) There exists $\gamma > 0$ such that

$$D_3f \leq -\gamma < 0 \quad \text{on} \quad G \times \langle -2r_0 - 1, 2r_0 + 1 \rangle \times \langle 0, \epsilon_0 \rangle,$$

where

$$r_0 > \gamma^{-1} \sup \{|f(t, x, 0, 0)|; (t, x) \in G\}.$$

3) $\sup\sup_{|u|^k \leq 1} \{|D^2f(t, x, u, \epsilon), (t, x) \in G, |u| \leq 2r_0 + 1, \epsilon \in \langle 0, \epsilon_0 \rangle\} < +\infty$.

Let $p_0 \in C^k$ be a solution of the equation $Qf[Zp, 0] = 0$ (which is unique). Then there exists $\epsilon^* > 0$ such that for $\epsilon \in \langle 0, \epsilon^* \rangle$ there exists a solution $u_\epsilon$ of the problem (5.1), (1.2) such that $u_0 = Zp_0$ and $u_\epsilon$ depends continuously on $\epsilon$ in the space $C_k$.

**References**


*Author's address: Praha 1, Žitná 25, ČSSR (Matematický ústav ČSAV).*