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A FAITHFUL CANONICAL REPRESENTATION FOR FINITELY GENERATED N-SEMIGROUPS

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Introduction. The term N-semigroup was first employed in [2] to denote a commutative, archimedean, cancellative, and non-potent semigroup. A commutative semigroup S is called archimedean if for any a, $\mathfrak{C} \in S$ there exist a positive integer m and an element $\mathfrak{C} \in S$ such that $a^m = \mathfrak{C}c$. By "non-potent" we mean "without idempotent". Such semigroups have been studied in papers [1], [2], [3], and [4]. In particular [4] develops a method for representing N-semigroups as the cartesian product of the additive non-negative integers and an abelian group, with a special operation defined on this product. This representation will be briefly outlined.

As defined in [4] an index function or I-function is a non-negative integer valued function defined on all ordered pairs (s, t) of the elements of an abelian group G. The index function satisfies the following:

1. $I(s, t) = I(t, s)$ $s, t \in G$.
2. $I(s, t) + I(st, r) = I(s, tr) + I(t, r)$ for all $s, t, r \in G$.
3. For any $s \in G$ there exists a positive integer m, which depends on s, that $I(s^m, s) > 0$.
4. $I(e, e) = 1$, where e is the identity of G.

Let J be the non-negative integers, let G be an abelian group, and let $I(s, t)$ be an I-function for G. The operation on $J \times G$ given by:

(1) $(i, s)(j, t) = (i + j + I(s, t), st)$ defines a N-semigroup on $J \times G$.

It is also shown in [4] that given any N-semigroup S, for any $a \in S$ there is an abelian group S_a^* and an index function I_a both uniquely determined by a, such that S is isomorphic to $J \times S_a^*$ (with I-function I_a). Clearly, there are many distinct such representations for a given S. It is also shown in [4] that the following is a partial ordering on any N-semigroup S:

For $x, y \in S$, $a \in S$ and a fixed, one says $x \underset{a}{<} y \Leftrightarrow x \neq y$ and there exists a positive integer n such that $x = a^n y$.

It is shown in [4] that the $\underset{a}{<}$ ordering satisfies the ascending chain condition. Elements maximal under the $\underset{a}{<}$ ordering are said to be *prime* to a.

In the following S is a finitely generated N-semigroup.

1. The $\underset{\Sigma}{>}$ Ordering. Definition 1.1. For $x, y \in S$ we have $x \underset{\Sigma}{>} y$ if and only if either there is $z \in S$ and $y = zx$ or $x = y$.

The following is useful.

Lemma 1.2. *Let $x, z \in T$ and N-semigroup. Then $x \neq xz$.*

Proof. Suppose $x = xz$, then by substitution we have $x = xz = (xz)z = x(zz)$ and cancellation gives $z = zz$. This contradicts the non-potent property of T.

Lemma 1.3. *$\underset{\Sigma}{>}$ is partial ordering of S.*

Proof. For $z, x, y \in S$ if $x \underset{\Sigma}{>} y$ and $y \underset{\Sigma}{>} z$ then $z = yz'$, $y = xz''$ and substitution gives $z = xz'z''$ and $x \underset{\Sigma}{>} z$. If $y \underset{\Sigma}{>} x$ and $x \underset{\Sigma}{>} y$ for sonne $x \neq y$ then $x = yz$, $y = xz'$ and $x = xzz'$ which is impossible by Lemma 1.2.

Lemma 1.4. *The $\underset{\Sigma}{>}$ ordering on S satisfies the ascending chain condition.*

Proof. Since S is finitely generated we may remove redundant elements from any finite generating set and obtain $\{a_1, a_2, \ldots, a_n\}$ as a minimal generating set. Suppose distinct x_i such that $x_1 \underset{\Sigma}{<} x_2 \underset{\Sigma}{<} x_3 \ldots$, let $x_1 = a_1^{k_1} \ldots a_n^{k_n}$, but $x_1 = x_2 z_2$, $z_2 \in S$ and $x_2 z_2 = a_1^{k_1} \ldots a_n^{k_n} = a_1^{(k_1'+k_1'')} \ldots a_n^{(k_n'+k_n'')}$ where $k_1' + k_1'' = k_1$ etc. and k_i' is the a_i exponent of x_2. Clearly some k_i has been reduced. Similarly $x_2 = x_3 z_3$ and since the k_i are finite and the a_i finite in number our chain must terminate.

Elements maximal in the $\underset{\Sigma}{\gtrless}$ ordering on S are called $\underset{\Sigma}{\gtrless}$ maximal elements. We may now show:

Theorem 1.5. *The $\underset{\Sigma}{\gtrless}$ maximal elements form a unique minimal generating set for S.*

Proof. Let $\{a_1, a_2, \ldots, a_n\}$ be any finite generating set for S. If $x \in S$ is $\underset{\Sigma}{\gtrless}$ maximal and $x \notin \{a_1, a_2, \ldots, a_n\}$ then since $x = \Pi a_i^{k_i}$ either some $k_i > 1$ or $k_i, k_j > 0$ for at least two i, j; $i \neq j$. In either case $x = a_i z$ for $z \in S$ which contradicts the definition of $\underset{\Sigma}{\gtrless}$ maximal element. On the other hand if some a_j is not $\underset{\Sigma}{\gtrless}$ maximal then we have $a_j = yz$ for $y, z \in S$. Expressing y, z in terms of the a_i we have:

$$a_j = a_1^{m_1} \ldots a_n^{m_n}.$$

If a_j fails to appear in the expression on the right we eliminate a_j from the generating set $\{a_i\}$. If a_j appears we have $a_j = a_j z$ which contradicts Lemma 1.2.

Corollary. *The $\underset{\Sigma}{>}$ maximal elements are maximal in any $\underset{\Sigma}{>}$ ordering as defined in the Introduction.*

2. Normal Standard Elements. The following is required.

Lemma 2.1. *S_a^* has finite order for any $a \in S$.*

Proof. Let $\{a_i, \ldots, a_n\}$ be a generating set for S. Select any $a \in S$, then $a = \Pi a_i^{k_i}$. It is shown in [4] that the order of S_a^* is equal to the number of elements in S prime to a. In [2] p. 10 it is shown that for any $x, y \in S$ there are positive integers m, n such that $x^m = y^n$. Thus, for all a_i in $\{a_1, \ldots, a_n\}$ there is a maximal positive integer n' such that $a_i^{n'}$ is not equal to a times some element of S. Thus, the number of elements prime to a in S is finite.

We may now make:

Definition 2.2. A normal standard element of S is any $a \in S$ such that S_a^* has minimal order.

That there are groups of minimal order is guaranteed by Lemma 2.1.

Definition 2.3. Let S_a^* and its corresponding I-function be a representation for S as defined in the introduction. Choose $x \in S$ and let x have representation (p, r) in terms of S_a^* and I_a. (i.e. $x = a^h r$, $h \geqq 0$, $r \in S_a^*$ (see [4]). We define $\mathfrak{J}(x)$ as:

$$\mathfrak{J}(x) = p|S_a^*| + \sum I_a(i, r)$$

as i ranges over S_a^*.

I am indebted to Professor Tamura for suggesting the following lemma.

Lemma 2.4. *For $x, y \in S$, where $x = (m, s)$, $y = (n, t)$ in terms of some S_a^* and its associated I_a, x is prime to y if and only if $m < I(t, t^{-1}s)$.*

Proof. Suppose $(p, r) \in S$ such that:

$(p, r)(n, t) = (p + n + I(r, t), rt) = (m, s)$. By definition we then have $r = t^{-1}s$ and $p + n + I(t, t^{-1}s) = m$. Thus, if $m < n + I(t, t^{-1}s)$, since p is always non-negative, no such (p, r) can exist.

If $m \geqq n + I(t, t^{-1}s)$ then choosing $p = m - (n + I(t, t^{-1}s))$ we have:

$(m - (n + I(t, t^{-1}s)), t^{-1}s)(n, t) = (m, s)$.

One then obtains:

Lemma 2.5. *For all $x \in S$, $\mathfrak{J}(x)$ is the number of elements of S prime to x.*

Proof. For any $x \in S$ with representation (m, s), x will be prime to $y \in S$, where $y = (n, t)$, when $m < n$. There are exactly $n|S_a^*|$ such elements, since by fixing m and letting n range through S_a^*, we obtain $|S_a^*|$ elements prime to (n, t). If $m \geqq n$ then $I(t, t^{-1}s) > 0$, by Lemma 2.4. Indeed, if $I_a(t, t^{-1}s) = k$, then we have (n, a),

$(n+1, a), \ldots, (n+k-1, a)$ and only these of the form (m, a), prime to (n, b). Thus the number of elements prime to (n, t) and where $m > n$ is just $\sum I(t, t^{-1}s)$, as s runs through all S_a^*, but this is just $\sum I(t, i)$ as i runs through all S_a^*.

Clearly, the normal standard elements of S are those for which $\mathfrak{J}(x)$ is minimal. To find such elements we may begin with any representation for S. We note that x is a normal standard element only if, when x is represented as (n, s), $n = 0$. Thus, if we construct a tabular representation of I_a for S_a^*, those elements $s \in S_a^*$ such that $\sum I_a(t, s)$ is minimal, as t ranges over S_a^*, will give normal standard elements in the form $(0, s)$. Practically, one examines the rows of the I_a table for rows with minimal sum, one then uses these group elements to form normal standard elements.

One may partially characterize normal standard elements by:

Theorem 2.5. *Every normal standard element is a $\underset{\Sigma}{>}$ maximal element.*

Proof. Let $x \in S$ be a normal standard element. Let us represent S by some S_a^* and its I_a. If $x = (0, r)$ in this representation and x is not $\underset{\Sigma}{>}$ maximal then $(0, r) = (0, s)(0, t)$, and from the definition of the operation S, $I_a(s, t) = 0$. Using property (2) of the definition of I-functions and summing over $i \in S_a^*$ we have: $\sum I(s, t) + \sum I(st, i) = \sum I(s, it) + \sum I(t, i)$. $I(s, t) = 0$ and thus: $\sum I(r, i) = \sum I(s, it) + \sum I(t, i)$. But $\sum I(r, i)$ is minimal and $\sum I(s, it) \geqq 1$ by property (3) of I-functions. This is clearly a contradiction.

In [2] PETRICH obtains a representation for N-semigroups with two generators. Using his terminology it is not difficult to show that an N-semigroup with two generators, in which $n_1 > n_2$, has two $\underset{\Sigma}{>}$ maximal elements but only one normal standard element. Thus, the converse of Theorem 2.5, is not true.

3. An Isomorphism Theorem. Let S, S' be two finitely generated N-semigroups. We then have the following.

Lemma 3.1. *Let the mapping $H: S \to S'$ be an isomorphism onto; then, if $a \in S$, is a normal standard element, $(a)H \in S'$ is a normal standard element of S', S_a^* is isomorphic to $S'^*_{(a)H}$ and I_a is identical to $I_{(a)H}$.*

Proof. $x \in S$ fails to be prime to a if and only if $x = y \,.\, a$, $x = ya$. But $(x)H = (y \,.\, a)H = (y)H\,(a)H$. This shows that the number of elements prime to a in S is not increased by a homomorphism. But an isomorphism onto implies an isomorphism H^{-1} from S' to S and normal standard elements are preserved. One now need only note that S_a^* and $S'^*_{(a)H}$ are defined by multiplication of elements of S and S' as follows. If $x, y \in S$ are prime to a then we may represent classes of S_a^* by x and y and $x \,.\, y$ (as elements of S) $= z \,.\, a^n$. But clearly $(x)H \,.\, (y)H = (z)H\,(a)H^n$. We now note that $I_a(x, y) = n$ the exponent of a in $x \,.\, y = z \,.\, a^n$. It is now clear that H preserves the structure of S_a^* and the values of I_a.

We may now show:

Theorem 3.2. *S is isomorphic onto S' if and only if S and S' have a common representation in terms of a structure group S^* and it is corresponding I-function.*

Proof. The only if portion of the above is immediate. But if S is isomorphic onto S' we may use Lemma 3.1 and any pair of normal standard elements a and $(a)H$ to obtain a common representation.

Thus, in the case of finitely generated N-semigroups the general problems of isomorphism discussed in [3] may be solved by examining the representations in terms of normal standard elements. This finite collection of representations may be used as a canonical set of representations. Then if one has two N-semigroup representations the method outlined in Section 2 may be used to construct the two sets of normal standard representations. If these two sets have a non-empty intersection then the two original N-semigroup representations really represent the same N-semigroup.

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