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OPENNESS OF LINEAR MAPPINGS IN *LF*-SPACES

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In a recent paper we have introduced and discussed the notion of orthogonality for pairs of subspaces of a locally convex space. There are many ways of characterizing this notion; these have been discussed at some length in [3]. For our purposes and for the particular case of Fréchet spaces, the following two will be sufficient.

Let Y and R be two closed subspaces of a Fréchet space E. We shall say that Y and R are *orthogonal* if one of the following two equivalent conditions is satisfied:

1° the natural mapping of $R \oplus Y$ onto $R + Y$ is open,

2° given two continuous linear functionals on Y and R which coincide on $R \cap Y$, there exists a continuous linear functional on E which is their common extension.

Clearly the relation of orthogonality is symmetric; we shall write simply $Y \perp R$ or $R \perp Y$.

In the present note we intend to introduce, for LF-spaces, a notion which describes the position of a subspace with respect to the spaces of a defining sequence. If R is a sequentially closed subspace of an LF-space F such that – roughly speaking – R is orthogonal to each element of a defining sequence for F, it turns out that R has many nice properties. In particular, this notion enables us to formulate a simple condition for a sequentially open mapping to be open. This condition – given in section two – is essentially equivalent to condition (4,4) of [2]. However, using the notion of orthogonality, both the statement and the proof of the theorem become exceedingly simple and transparent.

Some remarks concerning terminology and notation. If E is a locally convex space, we denote by $\mathbf{U}(E)$ the system of all closed absolutely convex neighbourhoods of zero in E. If Y is a subspace of E, we denote by $P(Y)$ the operator which assigns to each $x' \in E'$ its restriction to Y. If R is a subspace of an LF-space F, we say that R is sequentially closed in F if $x_n \in R$ and $\lim x_n = x$ implies $x \in R$. A continuous linear mapping of an LF-space E into an LF-space F is said to be sequentially open if its range is sequentially closed in F. For other equivalent descriptions of this notion see Proposition (3,2) of [2].

(1,1) Proposition. *If R is a sequentially closed subspace of an LF-space F, the following conditions are equivalent:*

1° *there exists a defining sequence F_j such that $F_j \perp (R \cap F_k)$ for each $k > j$;*

2° *for each defining sequence F_j there exists a sequence $p(j)$ of natural numbers, $p(j) \geqq j$, with the following property: given $k > p(j)$, $y' \in F'_{p(j)}$ and $r' \in (R \cap F_k)'$ which coincide on $R \cap F_{p(j)}$, there exists an $x' \in F'_k$ such that $P(F_j)\, x' = P(F_j)\, y'$ and $P(R \cap F_k)\, x' = r'$;*

3° *for each defining sequence F_j there exists another defining sequence $P_j \supset F_j$ such that $P_j \perp (R \cap P_k)$ for each $k > j$.*

If one of these conditions is fulfilled we shall say that R is orthogonal in F.

Proof. Assume 1° and consider a defining sequence H_j. Given j, there exist indices m, r such that

$$H_j \subset F_m \subset H_r .$$

Suppose that $k > r$ and consider a $y' \in H'_r$ and an $r' \in (R \cap H_k)'$ such that y' and r' coincide on $R \cap H_r$.

Take an $F_s \supset H_k$ so that $s > m$ and consider an extension p' of r' to $R \cap F_s$. Since $F_m \subset H_r$ the functionals $P(F_m)\, y'$ and p' coincide on $R \cap F_m$. Condition 1° being satisfied, there exists an $x' \in F'_s$ such that $P(F_m)\, x' = P(F_m)\, y'$ and

$$P(R \cap F_s)\, x' = p' .$$

Put $z' = P(H_k)\, x'$ so that $z' \in H'_k$. Since $H_k \subset F_s$, we have

$$\begin{aligned} P(R \cap H_k)\, z' &= P(R \cap H_k)\, P(H_k)\, x' = P(R \cap H_k)\, x' = \\ &= P(R \cap H_k)\, P(R \cap F_s)\, x' = P(R \cap H_k)\, p' = r' \end{aligned}$$

on the other hand, since $H_j \subset F_m \subset H_k$

$$\begin{aligned} P(H_j)\, z' &= P(H_j)\, P(H_k)\, x' = P(H_j)\, x' = P(H_j)\, P(F_m)\, x' = \\ &= P(H_j)\, P(F_m)\, y' = P(H_j)\, y' . \end{aligned}$$

Hence it suffices to take $p(j) = r$ and condition 2° is satisfied.

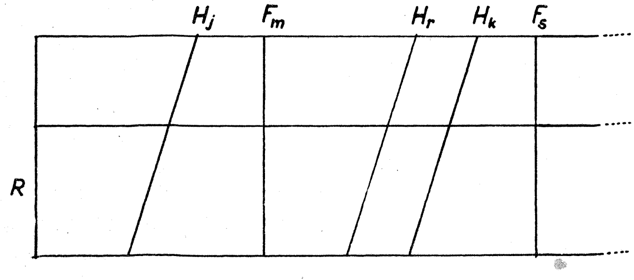

Fig. 1.

Suppose now that we are given a defining sequence F_j which satisfies condition 2°. Define first an increasing sequence of natural numbers $z(n)$, $n \in N$, as follows: $z(1) = 1$ and $z(n+1) = p(z(n)) + 1$. It follows that $z(n+1) > z(n)$ and $p(z(n+1)) \geqq z(n+1) > p(z(n))$. Define P_j as the closure in $F_{p(z(j))}$ of the space

$$F_{z(j)} + (R \cap F_{p(z(j))});$$

it follows that

$$F_{z(j)} \subset P_j \subset F_{p(z(j))} \tag{1}$$

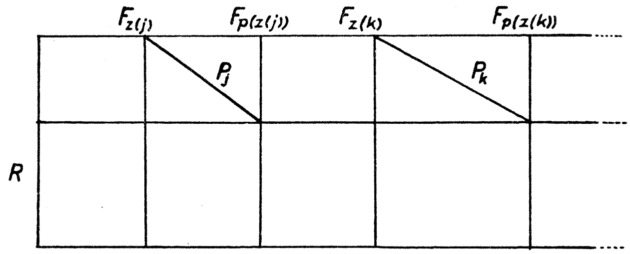

Fig. 2.

whence $R \cap P_j \subset R \cap F_{p(z(j))}$. Since, on the other hand, P_j contains, by its definition, the intersection $R \cap F_{p(z(j))}$, we have

$$R \cap P_j = R \cap F_{p(z(j))}. \tag{2}$$

Let $k > j$ and consider a pair of functionals $y' \in P'_j$ and $r' \in (R \cap P_k)'$ such that

$$P(R \cap P_j)\, y' = P(R \cap P_j)\, r'. \tag{3}$$

Since $P_j \subset F_{p(z(j))}$, there exists a $u' \subset F'_{p(z(j))}$ such that $P(P_j)\, u' = y'$. We have $p(z(k)) > p(z(j))$ and two functionals $u' \in F'_{p(z(j))}$ and $r' \in (R \cap P_k)' = (R \cap F_{p(z(k))})'$ which coincide on $R \cap P_j = R \cap F_{p(z(j))}$. It follows from condition 2° that there exists an $x' \in F'_{p(z(k))}$ with the following properties

$$P(F_{z(j)})\, x' = P(F_{z(j)})\, u' \tag{4}$$

$$P(R \cap F_{p(z(k))})\, x' = r'. \tag{5}$$

Since $P_k \subset F_{p(z(k))}$, we may form $z' = P(P_k)\, x' \in P'_k$. According to (4), we have

$$P(F_{z(j)})\, z' = P(F_{z(j)})\, P(P_k)\, x' = P(F_{z(j)})\, x' = P(F_{z(j)})\, u' =$$
$$= P(F_{z(j)})\, P(P_j)\, u' = P(F_{z(j)})\, y'.$$

It follows from (5) and (3) that

$$P(R \cap F_{p(z(j))})\, z' = P(R \cap F_{p(z(j))})\, x' = P(R \cap F_{p(z(j))})\, r' =$$
$$= P(R \cap P_j)\, r' = P(R \cap P_j)\, y' .$$

It follows that $z' \in P'_k$ coincides with y' on $F_{z(j)} + (R \cap F_{p(z(j))})$ and hence on P_j. Further, again by (5),

$$P(R \cap P_k)\, z' = P(R \cap F_{p(z(k))})\, x' = r' .$$

This completes the proof of the orthogonality of P_j and $(R \cap P_k)$. Since 3° implies 1° immediately, the proof is complete.

We shall need the following simple lemma.

(1,2) *Let H be an absolutely convex neighbourhood of zero in a locally convex space E. If $0 < \xi < 1$, then $\xi \bar{H} \subset H$.*

Proof. If $x \in \xi \bar{H}$, the set $x + (1 - \xi) H$ is a neighbourhood of x and hence intersects ξH. It follows that $x + (1 - \xi) h_1 = \xi h_2$ for suitable $h_1, h_2 \in H$. The set H being absolutely convex, $h_3 = -h_1 \in H$ so that $x = \xi h_2 + (1 - \xi) h_3 \in H$.

2. OPEN MAPPINGS

(2,1) Theorem. *Let E and F be two LF-spaces and let T be a continuous linear mapping of E into F. Suppose that the following two conditions are satisfied:*

1° *the mapping T is sequentially open,*

2° *the range of T is orthogonal in F.*

Then T is open.

Proof. Denote by R the range of T so that R is sequentially closed by proposition (3,2) of [2]. Let $U \in \mathbf{U}(E)$ and let us show that there exists a $V \in \mathbf{U}(F)$ such that $V \cap R \subset TU$. We shall apply lemma (4,2) of [2]. Since R is orthogonal in F there exists, by (1,1), a defining sequence F_j of F such that $F_j \perp R_k$ for each $k > j$ where $R_k = R \cap F_k$. Since T is sequentially open, there exists, by (3,2) of [2], for each n a $H_n \subset \mathbf{U}(F)$ such that $H_n \cap R_n \subset TU$. Suppose now we are given j, ε, V_j with the following properties: $0 < \varepsilon < 1$, $V_j \in \mathbf{U}(F)$ and $V_j \cap R_j \subset TU$. We intend to construct a $V_{j+1} \in \mathbf{U}(F)$ such that

(1) $$V_{j+1} \cap R_{j+1} \subset TU$$

(2) $$(1 - \varepsilon)(V_j \cap F_j) \subset V_{j+1} .$$

Take a $H_{j+1} \in \mathbf{U}(F)$ such that $H_{j+1} \cap R_{j+1} \subset TU$. Since $F_j \perp R_{j+1}$, the mapping

$F_j \oplus R_{j+1} \to F_j + R_{j+1}$ is open so that $(V_j \cap F_j) + (H_{j+1} \cap R_{j+1})$ is a neighbourhood of zero in $F_j + R_{j+1}$. Accordingly, there exists a $W \in \mathbf{U}(F)$ such that

(3) $$W \cap (F_j + R_{j+1}) \subset (V_j \cap F_j) + (H_{j+1} \cap R_{j+1}).$$

With $\sigma = \frac{1}{2}\varepsilon$ define

$$H = \operatorname{conv}((1 - \sigma)(V_j \cap F_j), \sigma W).$$

Let us show now that $H \cap R_{j+1} \subset TU$. Suppose that $p \in H \cap R_{j+1}$ so that p may be written in the form

$$p = \lambda\sigma w + (1 - \lambda)(1 - \sigma) z$$

with $0 \leqq \lambda \leqq 1$, $w \in W$, $z \in V_j \cap F_j$. If $\lambda = 0$, we have $p = (1 - \sigma) z \in V_j \cap F_j$ and $p \in R$ so that $p \in V_j \cap R_j \subset TU$. Hence we may suppose $\lambda > 0$. It follows that

$$\lambda\sigma w = -(1 - \lambda)(1 - \sigma) z + p \in F_j + R_{j+1}$$

whence $w \in F_j + R_{j+1}$ so that, according to (3), the vector w may be written in the form $w = z_0 + Tu_0$ for a suitable $z_0 \in V_j \cap F_j$ and $u_0 \in U$. Since $0 < \varepsilon < 1$, we have $0 < \sigma/(1 - \sigma) < 1$ so that $z_{00} = [\sigma/(1 - \sigma)] z_0 \in V_j \cap F_j$ as well. We have thus

$$p = (1 - \lambda)(1 - \sigma) z + \lambda\sigma z_0 + \lambda\sigma Tu_0 =$$
$$= (1 - \lambda)(1 - \sigma) z + \lambda(1 - \sigma) z_{00} + \lambda\sigma Tu_0 = (1 - \sigma) z_{000} + \lambda\sigma Tu_0$$

with $z_{000} \in V_j \cap F_j$. It follows that

$$z_{000} \in V_j \cap F_j \cap R \subset TU \text{ so that } z_{000} = Tu \text{ for some } u \in U.$$

Hence

$$p = T[(1 - \sigma) u + \sigma\lambda u_0] \in TU.$$

This proves the inclusion $H \cap R_{j+1} \subset TU$.

To sum up: we have constructed an absolutely convex set H with the following properties:

(4) H is a neighbourhood of zero if F (since $H \supset \sigma W$)

(5) $$(1 - \sigma)(V_j \cap F_j) \subset H.$$

(6) $$H \cap R_{j+1} \subset TU$$

We intend to show now that it suffices to take $V_{j+1} = \xi\overline{H}$ where $\xi = (1 - \varepsilon)/(1 - \sigma)$. First of all, the inclusion $V_{j+1} \in \mathbf{U}(F)$ is obvious. Since $V_{j+1} = \xi\overline{H} \subset H$, we have $V_{j+1} \cap R_{j+1} \subset H \cap R_{j+1} \subset TU$ and

$$(1 - \varepsilon)(V_j \cap F_j) = \xi(1 - \sigma)(V_j \cap F_j) \subset \xi H \subset \xi\overline{H} = V_{j+1}.$$

The proof is complete.

(2,2) Corollary. *Let F_j be an increasing sequence of Fréchet spaces such that the topology of F_{j+1} induces the topology of F_j on F_j. Let (F, u) be the inductive* *limit of the sequence F_j. Let R be a sequentially closed subspace of F. Let (R, v) be the inductive limit of the sequence $R \cap F_j$ so that v is finer than the restriction u_R of u to R. If R is orthogonal in F then $v = u_R$.*

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