Czechoslovak Mathematical Journal

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On a certain class of Λ-structures. II

Czechoslovak Mathematical Journal, Vol. 20 (1970), No. 1, 19–33

Persistent URL: `http://dml.cz/dmlcz/100941`

Czechoslovak Mathematical Journal, 20 (95) 1970, Praha

ON A CERTAIN CLASS OF Λ-STRUCTURES. II.

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(Received February 12, 1966, in revised form August 31, 1968)

This paper is a continuation of the previous article [17]. As to the terminology and notation, in the sequel we refer to [17] without any other comments. The numbering of theorems and definitions will be also preserved in conformity with [17]. Especially, the references to theorems of [17] will be indicated only by the corresponding number.

In Section 8 the completion of the investigated Λ-structure $(E(X), t)$ is characterized and the same question is discussed in some special cases for the unbounded topology t_0.

Section 9 and 10 is devoted to the applications of the previous results to the compactness in locally convex spaces. Theorem 17 presents a solution of a problem of V. PTÁK.

Two categories of locally convex spaces generalizing the investigated Λ-structures are introduced in Section 11 and 12. Further, there are established Extension Theorems of diverse types involving the recent results of A. GROTHENDIECK and V. PTÁK.

8. THE COMPLETION OF Λ-STRUCTURES

Any point x of X (or of $E(X)$) may be identified in the usual way with its canonical image $\hat{x}$ in $P^*(X)$. Moreover, $E(X)$ is a dense subspace in $P^*(X)$ under the weak topology. The same may be expected for the extended topology t in $P^*(X)$ of $\mathscr{H}$-convergence for a certain category of spaces X. In the next theorem we shall prove that for any uniform space X the completion $(\hat{E}(X), t)$ is a subspace of $(P^*(X), t)$. The main question to be solved in this section is the following one: to describe the category of all spaces X for which it holds

$$(\hat{E}(X), t) = P^*(X) .$$

Further, the completion of the Λ-structure $(E(X), t_{0c})$ will be investigated.

Theorem 12. *Let X be a uniform space. Then $(\hat{E}(X), t)$ is topologically isomorphic to a linear subspace of $(P^*(X), t)$.*

Proof. If z is a point of $(\hat{E}(X), t)$ then (cf. [5]) z is a linear function on $P(X)$ continuous on each $H \in \mathscr{H}$ in the pointwise topology $\sigma(P(X), E(X))$. To prove $z \in P^*(X)$ we consider a sequence $\{f_n\}$ in the Banach algebra $P(X)$ such that $\lim f_n = 0$. The absolutely convex envelope of all f_n, $n = 1, 2, \ldots$ is an element of $\mathscr{H}$. Since $f_n \to 0$ in the pointwise topology, we have $\lim \langle f_n, z \rangle = 0$, hence $z \in P^*(X)$.

Remark. The assertion of Theorem 12 follows directly from the fact that $(P^*(X), t)$ is complete for every uniform space X.

Proof. Let $\mathscr{F}$ be a Cauchy filter in $(P^*(X), t)$. If f_0 is a function defined for each $\xi \in P(X)$ by $\lim \mathscr{F}(\xi) = \langle \xi, f_0 \rangle$, then obviously f_0 is a linear function continuous on each $H \in \mathscr{H}$. Since every sequence $\{f_n\}$, $f_n \to 0$ in $P(X)$, belongs to $\mathscr{H}$, we obtain $\langle f_n, f_0 \rangle \to 0$, hence $f_0 \in P^*(X)$.

If now X is precompact then according to Theorem 2 any $z \in P^*(X)$ is continuous on each $H \in \mathscr{H}$ in the pointwise topology, consequently $z \in (\hat{E}(X), t)$. But $P(X)$ is isometric to the space $C(\mathfrak{M})$ of all continuous functions on the compact space $\mathfrak{M}$ of all maximal ideals of $P(X)$, hence we have proved

Theorem 13.[1]) *Let X be a precompact space, $\mathfrak{M}$ the space of all maximal ideals of the Banach algebra $P(X)$. Then the completion $(\hat{E}(X), t)$ is algebraically isomorphic to the space of all Radon measures on $\mathfrak{M}$.*

Corollary. *If X is pseudocompact then $(\hat{E}(X), t_c)$ is algebraically isomorphic to $C^*(X)$.*

The last assertion generalizes a theorem of M. Katětov (cf. [10], [15]).

To elucidate the role of the extended topology t on $P^*(X)$, we note that for any precompact space X this topology is compatible with the duality of the dual pair $\langle P(X), P^*(X) \rangle$, particularly any $H \in \mathscr{H}$ is relatively weakly compact.

Lemma. *Let X be a uniform space, $\hat{X}$ the closure of X in $(\hat{E}(X), t)$ and $\bar{X}^\sigma$ the closure of X in $P^*(X)$ under the weak topology. Then the following statements are equivalent:*

(a) *X is precompact.*

(b) *It holds $\hat{X} = \bar{X}^\sigma$.*

Proof. (a) $\Rightarrow$ (b). If X is precompact, then according to Theorem 2 we have $t = \sigma$ on X, hence $\bar{X}^\sigma$ is the completion of $(X, \sigma) = (X, t)$.

(b) $\Rightarrow$ (a). Suppose that (b) holds. Since t is the uniformity on $\hat{X}$ of $\mathscr{H}(\hat{X})$-convergence and σ the uniformity on $\hat{X}$ of $P(\hat{X})$-convergence, they induce the same topology on $\hat{X} = \bar{X}^\sigma$. But $\bar{X}^\sigma$ under the topology σ is a compact space, hence $(\hat{X}, t)$ is compact.

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[1]) This theorem has been proved independently for compact space by K. John.

If X is a non-precompact space, then we obtain $\overline{X}^{\sigma} \neq \hat{X}$. From

$$\hat{X} = \overline{X}^{\sigma} \cap (E(X), t)$$

(cf. [17], formula (4)) and from Theorem 8 we now obtain $(\hat{E}(X), t) \neq P^*(X)$. This implies

Theorem 14. *Let X be a uniform space, ω_0 the canonical topological isomorphism of $(\hat{E}(X), t)$ into $(P^*(X), t)$ defined by Theorem 12. The following statements are equivalent:*

(a) *X is precompact.*

(b) *ω_0 is a topological isomorphism of $(\hat{E}(X), t)$ onto $(P^*(X), t)$.*

(c) *The topology t in $P^*(X)$ is compatible with the duality of the dual pair $\langle P(X), P^*(X)\rangle$.*

If X is a completely regular non-pseudocompact space, then we have $t_c \neq \sigma$ on X, hence (X, t_c) is non-precompact. Thus we have obtained

Theorem 15. *Let X be a completely regular space, ω_0 the canonical isomorphism of $(\hat{E}(X), t_c)$ into $C^*(X)$. Then the following statements are equivalent:*

(a) *X is pseudocompact.*

(b) *The mapping ω_0 is a topological isomorphism of $(\hat{E}(X), t_c)$ onto $(C^*(X), t_c)$.*

(c) *The topology t_c on $C^*(X)$ is compatible with the duality of the dual pair $\langle C(X), C^*(X)\rangle$.*

Now we intend to describe the completion of the Λ-structure $(E(X), t_0)$ over a certain category of completely regular spaces. First we consider the space $C(X)$ with the locally convex topology $k(X)$ of the compact convergence in X. Suppose that p is the topology in $(E(X), t_{0c})^*$ of the precompact convergence on $(E(X), t_{0c})$; obviously we have

$$\sigma(C_0(X), E(X)) \leqq k(X) \leqq p .$$

Hence, on each $M \in \mathscr{M}_0$ the topologies $k(X)$ and $\sigma(C_0(X), E(X))$ coincide. If f is now in $C_0^*(X)$ then f is continuous in the pointwise topology on each $M \in \mathscr{M}_0$, consequently $f \in (\hat{E}(X), t_{0c})$. Thus for any completely regular space X the inclusion $C_0^*(X) \subseteq (\hat{E}(X), t_{0c})$ holds.

Let us denote by $\overline{C}_0^*(X)$ the space of all linear and sequentially continuous functions on $C_0(X)$. Obviously $C_0^*(X)$ is a subset of $\overline{C}_0^*(X)$ and we have

Theorem 16. (a) *If X is a completely regular space then $C_0^*(X) \subseteq (\hat{E}(X), t_{0c})$.*

(b) *For any locally compact space X it holds*

$$C_0^*(X) \subseteq (\hat{E}(X), t_{0c}) \subseteq \overline{C}_0^*(X) .$$

(c) *Let X contain a dense countable subset, then*

$$\bar{C}_0^*(X) \subseteq (\hat{E}(X), t_{0c}) .$$

(d) *For any locally compact and countable at the infinity space X the completion $(\hat{E}(X), t_{0c})$ is identical with $C^*(X)$.*

Proof. The statement (a) was already proved. To establish (b) we consider a sequence $\{f_n\}$ in $C_0(X)$, $f_n \to 0$ in the topology of the compact convergence. It is easy to see that the family $\{f_n\}$ lies in $\mathcal{M}_0$, hence $\lim \langle z, f_n \rangle = 0$ for any $z \in (\hat{E}(X), t_{0c})$. If the condition of (d) is satisfied, then $C_0(X)$ is metrizable, consequently $C_0^*(X) = \bar{C}_0^*(X)$.

The statement (c) follows directly from

Lemma. *Let X be a completely regular space with a dense countable subset. Then any $M \in \mathcal{M}_0$ is a metrizable subspace of $C_0(X)$.*

Proof. If X_1 is a dense countable subset of X, then $E(X_1)$ is dense in $(E(X), t_{0c})$. Similarly the countable subset of all $\sum r_i x_i$, r_i rational, $x_i \in X_1$, is dense in $(E(X), t_{0c})$, hence (cf. [1]) any $M \in \mathcal{M}_0$ is metrizable in the pointwise topology. The assertion of Lemma follows now from the above mentioned relation $k(X) = \sigma(C_0(X), E(X))$.

Remark 1. As it was stated in Section 5 the space $(E(X), t)$ is not sequentially complete for any uniform space X, hence $(E(X), t) \neq (\hat{E}(X), t)$. It should be also noticed that $(E(X), t)$ does not posses the property of semi-reflexivity. In the example that follows we shall show that these statements need not be true for the unbounded topology t_0.

Example. Let $X = N$ be the collection of all integers with the discrete topology. The space $C_0(N) = (E(N), t_{0c})^*$ consists then of all sequences of real numbers and under the topology of compact convergence it may be identified with the strong topological dual of $(E(N), t_{0c})$ (see Theorem 4 of [17]). But it is well-known that such space is topologically isomorphic with the Cartesian product ΠR_n, R_n being the real line for all n, with the usual topology. Since the strong topological dual of ΠR_n coincides with the corresponding direct sum $\sum R_n = E(N)$, we have established the semi-reflexivity of $(E(N), t_{0c})$. Now we shall prove, more generally, the reflexivity of $(E(N), t_{0c})$. To this point it suffices to note that t_{0c} is the finest locally convex topology on $E(N)$. Evidently for the last topology τ_ω it holds $\tau_\omega \geqq t_{0c}$. But τ_ω induces on N a discrete topology; consequently the canonical embedding of $(E(N), t_{0c})$ onto $(E(N), \tau_\omega)$ is continuous, hence $t_{0c} \geqq \tau_\omega$.

From Theorem 16, (d) we can conclude also the completeness of $(E(X), t_{0c})$.

Remark 2. The statement (d) of Theorem 16 admits a descriptive interpretation. If X is a locally compact and countable at the infinity space then $(\hat{E}(X), t_{0c})$ consists of all Radon measures with the compact support on X. The verification of this assertion may be found in [2].

Remark 3. Similarly, if X is pseudocompact, then the family of all positive elements $z \in (\hat{E}(X), t_c)$ coincides with the collection of all Daniell integrals on X. Indeed, if $f_n \in C(X)$, $f_n \downarrow 0$, then evidently $f_n \to 0$ in the space $C(X)$. For any $z \in (\hat{E}(X), t)$ we obtain $\langle z, f_n \rangle \to 0$. On the other hand, if F is Daniell integral on $X, f_n \to 0$ in the space $C(X)$, then $\varepsilon_n \downarrow 0$ where

$$c_n = \max_{x \in X} |f_n(x)|, \quad \varepsilon_n = \max(c_n, c_{n+1}, \ldots)$$

If e is now the unit element of the algebra $C(X)$ then eividently $-\varepsilon_n . e \leqq f_n \leqq \varepsilon_n . e$ and $-\varepsilon_n F(e) \leqq F(f_n) \leqq \varepsilon_n F(e)$. Hence $\lim F(f_n) = 0$ and, consequently, $F \in C^*(X)$.

9. A DUAL CHARACTERIZATION OF PSEUDOCOMPACT SPACES

Now we consider $E(X)$ endowed with the Mackey topology $\tau = \tau\langle E(X), C(X) \rangle$. In this section we shall be concerned with a question presented by M. Katětov: to describe the completion of the Λ-structure $(E(X), \tau)$. As a consequence we intend to show (see also Section 10) some interesting applications to the compactness in locally convex spaces.

Denote by $\mathscr{K}(X)$ the family of all relatively weakly compact subsets in $C(X)$, by $\mathscr{K}_0(X)$ the system of all bounded and relatively compact subsets in $C(X)$ with the pointwise topology $\sigma(C(X), X)$. First we recall (cf. [13], [14]) that for a pseudocompact space X it holds $\mathscr{K}(X) = \mathscr{K}_0(X)$. Previously this statement has been proved for countably compact spaces by integration methods in [7]; a quite elementary proof (using the combinatorial lemma) for pseudocompact spaces is given in [12], [13]. To this point let us note that the case of a pseudocompact space may be immediately reduced by Theorem 34 of [9] or by some of its consequences (e.g. Lemma 1.1 of [14]) to the case that X is compact.

Lemma. *If τ_0 is the Mackey topology on $C^*(X)$ defined by the dual pair $\langle C^*(X), C(X) \rangle$, then $(C^*(X), \tau_0)$ is complete for any completely regular space X.*

Proof. For any sequence $\{f_n\}$ in $C(X), f_n \to 0$, the family $\{f_n\}$ is uniformly bounded and compact in $C(X)$, hence it is weakly compact. The rest of the proof may be carried out as in the remark to Theorem 12.

The completion $(\hat{E}(X), \tau)$ of the Λ-structure $(E(X), \tau)$ is, similarly as in Section 8, algebraically isomorphic to a subspace of $C^*(X)$ for any completely regular space X. In view of this isomorphism (denoted by ω_0) we may, of course, identify $(\hat{E}(X), \tau)$ with its ω_0-image in $C^*(X)$. To prove this statement, we consider first the identical algebraical embedding J of $(E(X), \tau)$ onto $(E(X), t_c)$. By Theorem 4, § 18 of [11], there exists a uniquely defined extension $\tilde{J}$ of $(\hat{E}(X), \tau)$ into $(\hat{E}(X), t_c)$, but $(\hat{E}(X), t_c)$ is algebraically isomorphic to a subspace of $C^*(X)$ in accordance with Theorem 12.

The main result of this section is contained in (see also [16])

Theorem 17. *Let X be a completely regular space. Then the following statements are equivalent*:

(a) *X is pseudocompact.*

(b) *It holds $\mathscr{K}(X) = \mathscr{K}_0(X)$.*

(c) *The canonical mapping ω_0 is a topological isomorphism of $(\hat{E}(X), \tau)$ onto $(C^*(X), \tau)$.*

Proof. (a) $\Rightarrow$ (b): it follows from the well-known theorem mentioned above.
(b) $\Rightarrow$ (c): if (b) holds, then $(E(X), \tau)$ is a dense subspace of $(C^*(X), \tau)$. Indeed, the assertion (b) says that the Mackey's topologies defined by the dual pairs $\langle E(X), C(X)\rangle$ and $\langle C^*(X), C(X)\rangle$ coincide. The rest follows from the previous Lemma.
(c) $\Rightarrow$ (a): it is a consequence of the relation $(\hat{E}(X), \tau) \subseteq (\hat{E}(X), t_c)$ and of Theorem 15.

It should be noticed that the implication (b) $\Rightarrow$ (a) is a solution of the problem presented by V. Pták (cf. [14]).

10. SOME APPLICATIONS TO THE COMPACTNESS

In the present section we shall prove, as applications of the previous results, some theorems concerning the compactness in locally convex spaces.

In the substance, the method we are going to make use of may be elucidated by a simple example which follows. Let X be a bounded subset of a Banach space F. Putting $h(z) = \sum_{x \in X} \lambda(x)\, x$ for any $z = \{\lambda(x), x \in X\} \in l^1(X)$, we define a continuous linear operator of $l^1(X)$ into F. Such situation turns out, especially, if we intend to characterize the completion of the (projective) topological tensor product $E \hat{\otimes} F$ of two Banach spaces (cf. [8], § 2). In the stated example we may proceed, notwithstanding, as follows: first the subset X is endowed with a special discrete topology ϱ and then the identical (and continuous) mapping h is extended to the completion $(\hat{E}(X), \varrho)$ (compare with Theorem 7). The proofs of the following theorems are based on the same idea with the only difference that X will be considered to be a topological subspace of F.

If, moreover, X is compact in the above-mentioned example, then h is a compact operator on $l^1(X)$ (cf. [3], Ch. III, § 3). The last statement suggests somewhat different point of view, namely, it suggests to apply the properties of compact operators (cf. [6]), expecially if we take into account that for a compact subset X the bidual space $(E(X), t)^{**}$ coincides with $C^*(X)$ (see Theorem 1). For Banach spaces the theorem of KREIN and of EBERLEIN have been proved by a method which depends basically on the properties of compact operators and on the Extension Theorem in [13].

Suppose that X is a subset of a locally convex space F, ΓX its absolutely convex envelope in F, U the absolutely convex envelope of X in $E(X)$. If J is the identical embedding of X into F and $\tilde{J}$ its linear extension to $E(X)$, then it is clear that $\tilde{J}(X) = X$ and $\tilde{J}(U) = \Gamma X$. For the Mackey topology in F we write $\tau_F = \tau(F, F^*)$.

The first theorem we are going to prove is the KREIN theorem in the general form (cf. [7]).

Theorem 18. *Let X be a weakly relatively compact subset in a locally convex space F. Suppose that the closed and absolutely convex envelope $\overline{\Gamma X}$ in F is complete in F under the Mackey topology τ_F. Then $\overline{\Gamma X}$ is weakly compact in F.*

Proof. Without loss of generality it suffices to prove the theorem under the condition that X is weakly compact in F (according to the equality $\overline{\Gamma X} = \overline{\Gamma \overline{X}}$ we take $\overline{X}$ if necessary). If X is weakly compact then we consider the uniformity t_c induced on X by the weak topology $\sigma = \sigma(F, F^*)$ in F. The canonical embedding $J : X \to (F, \sigma)$ is now bounded and continuous, hence it may be continuously extended to $(E(X), t_c)$. From the well-known properties of linear continuous mappings in locally convex spaces (cf. [1]) it follows that the linear extension $\tilde{J}$ of J is continuous in the topologies τ and τ_F where $\tau = \tau(E(X), C(X))$. With respect to Theorem 17 we may regard $\tilde{J}$ as a continuous mapping from $(C^*(X), \tau)$ into the completion $\hat{F}$ of (F, τ_F). But $\overline{\Gamma X}$ being τ_F-complete in F, it is closed in $\hat{F}$. For the weak closure $\overline{U}^\sigma$ of U in $C^*(X)$ it follows $\tilde{J}(\overline{U}^\sigma) \subseteq \overline{\Gamma X}$. On the other hand, $\overline{U}^\sigma$ is weakly compact and absolutely closed in $C^*(X)$, hence $\tilde{J}(\overline{U}^\sigma)$ is weakly compact in $\hat{F}$ and contains $\overline{\Gamma X}$. This implies the equality

$$\tilde{J}(\overline{U}^\sigma) = \overline{\Gamma X} .$$

The proof is complete.

Theorem 19. *Suppose that X is a subset of a locally convex space F. If the closed and convex envelope $\overline{\mathrm{co}\, X}$ of X is complete for the Mackey topology τ_F, then the following properties are equivalent:*

(a) *X is relatively compact in F.*

(b) *X is relatively countable compact in F.*

(c) *X is relatively pseudocompact in F.*

Proof. Obviously (a) $\Rightarrow$ (b) $\Rightarrow$ (c). We shall prove (c) $\Rightarrow$ (a). Similarly as in the preceding proof we may suppose that X is pseudocompact in F. Let us consider the topology t_c induced on X by the original topology in F. The closure $\overline{X}$ in $(\hat{E}(X), t_c)$ is therefore compact and by the same reasoning as in the proof of Theorem 18 we may establish the continuity of the canonical mapping $\tilde{J} : (C^*(X), \tau) \to (F, \tau_F)$. Because of the relation $\tau \geqq t_c$, the set $\bar{X}$ is closed in the topology τ in $C^*(X)$. From $\tilde{J}(\bar{X}) \subseteq \subseteq \overline{\text{co}\, X} \subseteq F$ we conclude that $\tilde{J}(\bar{X})$ is compact in F. Evidently the closure of X in F is contained in $\tilde{J}(\bar{X})$. This proves the assertion.

Remark 1. The assertion of the last theorem generalizes simultaneously the EBERLEIN-GROTHENDIECK theorem (cf. [7], Prop. 2) and the theorem of V. PTÁK (cf. [14]). It is to be noted that the preceding statement may be derived by the procedure indicated in [7] from the mentioned result of V. PTÁK.

Remark 2. The method of Λ-structures over a completely regular space X regarding $C(X)$ as a dual to $E(X)$ (with a suitable topology) makes it possible to investigate also other concepts of compactness in $C(X)$.

Let X be, for example, a completely regular space, M a convex subset in $C(X)$. Suppose that M is convex-compact in the pointwise topology σ in $C(X)$ (i.e. the intersection of any decreasing sequence $\{K_i\}$ of non-empty, convex and closed subsets in M in the pointwise topology is non-empty). If h is an arbitrary function on X adherent to M in the pointwise topology, then h is sequentially continuous on X.

Proof. From the assumption it follows immediately that h is a point of the algebraical dual space to $(C(X), \sigma)^*$. For any sequence $\{x_i\}$ in X there exists $h_0 \in M$ such that (cf. [11])

$$\lim \left[h(x_i) - h_0(x_i)\right] = 0 .$$

If now $\{x_i\}$ is a sequence in X, $x_i \to x_0 \in X$ then we choose h_0 in M having the mentioned property with respect to the sequence $x_1, x_0, x_2, x_0, \ldots, x_n, x_0, \ldots$ Evidently it holds

$$\lim h(x_i) = \lim h_0(x_i) = h_0(x_0) = h(x_0) .$$

It should be noticed that the continuity of h has been proved under the same conditions as above and for the compact space X in [4].

11. AN EXTENSION THEOREM FOR $\mathcal{DB}$-SPACES

Let E and F be two locally convex spaces, $B(x, y)$ a bilinear function on the Cartesian product $E \times F$. The linear mappings u and v defined by the formula

$$B(x, y) = \langle x, u(y) \rangle = \langle v(x), y \rangle ,$$

$x \in E$, $y \in F$ obviously map F into the algebraical dual to E and E into the algebraical dual space to F. If B is, moreover, separately continuous, then u is a continuous mapping of F into E^* under the weak topology and a similar statement holds for the mapping v. Because of the relation $E \subseteq E^{**}$ and $F \subseteq F^{**}$ a natural question arises under what conditions is it possible to extend in a unique manner such a separately continuous bilinear function $B(x, y)$ to the Cartesian product $E^{**} \times F^{**}$. In [8] there are indicated some strong conditions under which such an extension exists. From a different point of view V. PTÁK (cf. [13]) has solved the question under what conditions there exists a uniquely defined extension of a separately continuous function $B(x, y)$ on the Cartesian product $X \times Y$ of two completely regular spaces to the product $C^*(X) \times C^*(Y)$.

To generalize these uncomparable standpoints, we shall introduce the concept of a $\mathscr{DB}$-space and then we shall prove the extension theorem for the Cartesian product of two $\mathscr{DB}$-spaces.

Definition 3. A locally convex space E is said to be a $\mathscr{DB}$-space if the following properties are satisfied:

1° There exists a bounded subset X in E such that $\{nX; n = 1, 2, \ldots\}$ forms the fundamental sequence of bounded subsets in E.

2° A linear operator u of E into an arbitrary locally convex space G is continuous if and only if the restriction of u to X is continuous in the induced topology.

It is easy to see that any normed space and the Λ-structure $(E(X), t)$ are $\mathscr{DB}$-spaces; the strong dual of a $\mathscr{DB}$-space is Banach space.

In further discussion we shall suppose that $B(x, y)$ is a separately continuous bilinear real-valued function on the Cartesian product $E \times F$ of two $\mathscr{DB}$-spaces $E = (E, X)$, $F = (F, Y)$ where X and Y are forming the fundamental system of bounded subsets in E and F in accordance with the property 1° of Definition 3.

Lemma 1. *If $u(Y)$ is relatively $\sigma(E^*, E^{**})$-compact in E, then*

(a) *u is a continuous mapping of F into E^* under the topologies $\sigma(F, F^*)$ and $\sigma(E^*, E^{**})$;*

(b) *the adjoint mapping ${}^t u : E^{**} \to F^*$ is continuous in the topologies $\sigma(E^{**}, E^*)$ and $\sigma(F^*, F)$.*

Proof. It suffices to prove the first assertion. The space E being a dense subspace in E^{**} in the topology $\tau(E^{**}, E^*)$ and $\Gamma\, u(Y)$ being relatively $\sigma(E^*, E^{**})$-compact, there exists for any $x' \in E^{**}$ and each $\varepsilon > 0$ an $x \in E$ such that

$$|\langle x' - x, \Gamma\, u(Y)\rangle| < \tfrac{1}{2}\varepsilon \,. \tag{8}$$

Let W be a weak neighbourhood in E^* defined by the formula

$$W = \{f \in E^*;\ |\langle f, x_i'\rangle| < \varepsilon,\ 1 \leqq i \leqq n\}$$

where $x_i' \in E^{**}$ for $1 \leqq i \leqq n$. For any such x_i' we may find some $x_i \in E$ satisfying (8). Putting

$$V = \{y \in F;\ |B(x_i, y)| < \tfrac{1}{2}\varepsilon,\ 1 \leqq i \leqq n\}$$

we see that $y \in V \cap \Gamma Y$ implies

$$|\langle x_i', u(y)\rangle| \leqq |\langle x_i' - x_i, u(y)\rangle| + |\langle x_i, u(y)\rangle| < \varepsilon .$$

From the last relation we may conclude that u is continuous on Y, hence u is continuous on F in the required topologies.

Lemma 2. *If $B(x, y)$ is a separately continuous bilinear real-valued function on $E \times F$, then the following properties are equivalent:*

1° *$u(Y)$ is relatively $\sigma(E^*, E^{**})$-compact in E.*

2° *$v(X)$ is relatively $\sigma(F^*, F^{**})$-compact in F.*

Proof. Since the absolutely convex envelope $A = \Gamma\, u(Y)$ is weakly relatively compact in E^*, the canonical bilinear function $\langle f, x'\rangle$ satisfies the double limit condition on $A \times U^0$, U^0 being the unit ball in E^{**}. It is easy to see that the same property is satisfied for the function

$$\langle x, u(y)\rangle = \langle v(x), y\rangle$$

on $X \times \Gamma Y$. Evidently $v(X)$ is bounded in F^* and ΓX is a dense subset of the unit ball in F^*. Hence, $v(X)$ is relatively weakly compact.

A separately continuous bilinear function on $E \times F$ satisfying the property 1° (or equivalently 2°) of Lemma 2 is said to be *weakly compact* (cf. [8]).

From the preceding lemma we may, in particular, conclude that ${}^t v(F^{**}) \subseteq E^*$.

Lemma 3. *If B is a weakly compact bilinear function on $E \times F$, then ${}^t u$ is continuous in the topologies $\sigma(E^{**}, E^*)$ and $\sigma(F^*, F^{**})$. A symmetrical assertion holds for the operator ${}^t v$.*

Proof. It suffices to prove

$$\langle {}^t u(x_0), y_0\rangle = \langle x_0, {}^t v(y_0)\rangle$$

for all $x_0 \in E^{**}$, $y_0 \in F^{**}$ (compare with [8] and [13]). For the sake of simplicity we shall assume that x_0 and y_0 are points of the unit balls in E^{**} and F^{**}. For any $\varepsilon > 0$ we choose $x_1 \in E$ such that

$$|\langle x_1 - x_0, {}^t v(y_0)\rangle| < \tfrac{1}{2}\varepsilon$$

and

$$|\langle {}^t u(x_1 - x_0), y\rangle| = |\langle x_1 - x_0, u(y)\rangle| < \tfrac{1}{2}\varepsilon$$

for all $y \in Y$. This implies

$$|\langle {}^t u(x_0), y_0 \rangle - \langle x_0, {}^t v(y_0) \rangle| \leqq |\langle {}^t u(x_0), y_0 \rangle - \langle x_1, {}^t v(y_0) \rangle| + \\ + |\langle x_1, {}^t v(y_0) \rangle - \langle x_0, {}^t v(y_0) \rangle| < \varepsilon .$$

The proof is complete.

Now we are ready to establish Extension Theorem for $\mathcal{DB}$-spaces.

Theorem 20. *Let $B(x, y)$ be a separately continuous bilinear real-valued function on the Cartesian product $E \times F$ of two $\mathcal{DB}$-spaces. Then the following properties are equivalent:*

(a) *B is a weakly compact bilinear function.*

(b) *B has a bilinear separately continuous extension to $E^{**} \times F^{**}$ in the weak topologies.*

Proof. (a) $\Rightarrow$ (b): we put

$$B(x, y) = \langle {}^t u(x), y \rangle = \langle x, {}^t v(y) \rangle$$

for all $x \in E^{**}$ and $y \in F^{**}$.

(b) $\Rightarrow$ (a): is evident.

If $B(x, y)$ is a function on the Cartesian product $X \times Y$ of two uniform spaces, then we define uniquely the bilinear extension $\tilde{B}$ to $E(X) \times E(Y)$ by the formula

$$\tilde{B}(x, y) = \sum \lambda_i \mu_j B(x_i, y_j)$$

where $x = \sum \lambda_i x_i \in E(X)$, $y = \sum \mu_j y_j \in E(Y)$. Let B be a bounded and separately uniformly continuous function on $X \times Y$. Then $\tilde{B}$ is evidently separately continuous on $(E(X), t) \times (E(Y), t)$. From the last theorem we may now conclude

Theorem 21 (Pták). *Let $B(x, y)$ be a bounded and separately uniformly continuous function on the Cartesian product $X \times Y$ of two uniform spaces. Then the following assertions are equivalent:*

(a) *$u(Y)$ is relatively weakly compact in $P(X)$.*

(b) *$v(X)$ is relatively weakly compact in $P(Y)$.*

(c) *B possesses a bilinear and separately continuous extension to $P^*(X) \times P^*(Y)$ in the weak topologies.*

If $B(x, y)$ is a bounded and uniformly continuous function on the Cartesian product $X \times Y$ of two precompact spaces, then evidently B satisfies the property (a) of Theorem 21. Moreover, the subset $u(Y)$ being relatively compact in $P(X)$, we may expect that somewhat stronger results can be established in this case. It would be of some interest to investigate also functions satisfying weaker conditions than uniform continuity, for example, (uniform) hypocontinuity on $X \times Y$. Partial results of this kind were communicated in [18]. As to the assumptions of the precompactness we note only that for any precompact X the space $(P^*(X), t)$ represents the Grothendieck's bidual space to $(E(X), t)$ with the natural topology.

In the following theorem we shall be concerned with a simultaneous extension of a family $\{B_\alpha\}$, α varying over a set A of indices. For any $\alpha \in A$ the mappings u_α, v_α are defined by the usual formula

$$B_\alpha(x, y) = \langle x, u_\alpha(y) \rangle = \langle v_\alpha(x), y \rangle .$$

Theorem 22. *Let X and Y be two precompact spaces, $\{B_\alpha;\ \alpha \in A\}$ a uniformly bounded family of uniformly continuous functions on $X \times Y$. Then the following properties are equivalent:*

(a) *The family $\{B_\alpha;\ \alpha \in A\}$ is uniformly equicontinuous on $X \times Y$.*

(b) *The family $\{B_\alpha;\ \alpha \in A\}$ is uniformly hypoequicontinuous on $(P^*(X), t) \times \times (P^*(Y), t)$ (with respect to the collections of all bounded subsets in $P^*(X)$ and in $P^*(Y)$).*

(c) *The family $\{B_\alpha;\ \alpha \in A\}$ is equicontinuous on each $P^*(X) \times N$, N bounded in $P^*(Y)$, in the topology induced by the space $(P^*(Y), t) \times (P^*(X), \sigma)$. The symmetrical assertion holds as well.*

(d) *The family $\{B_\alpha;\ \alpha \in A\}$ is weakly equicontinuous on each $M \times N$, M bounded in $P^*(X)$, N bounded in $P^*(Y)$.*

The proof may be carried out making use of Theorem 5; it is easy to see that the family $\{u_\alpha;\ \alpha \in A\}$ may be extended uniformly equicontinuously to the family $\{\tilde{u}_\alpha;\ \alpha \in A\}$ where $\tilde{u}_\alpha : (E(Y), t) \to P(X)$.

12. THE UNBOUNDED TOPOLOGY

In what follows we shall present a generalization of the Λ-structure $(E(X), t_{0c})$ and, further, some results of the preceding sections will be extended to a more general category of spaces.

Recall that $C_0(X)$ is the space of all continuous functions on X with the topology of compact convergence.

Definition 4. Let $\mathscr{B} = \{B_\alpha;\ \alpha \in \Omega\}$ be a system of bounded subsets of a locally convex space E. The space E will be called a $\mathscr{D}\mathscr{M}$-space (denoted $(E, \mathscr{B})$) if

1° for any bounded subset B in E there exists a suitable $\alpha \in \Omega$ and an integer n such that $B \subseteq n\, \Gamma B_\alpha$;

2° a linear mapping u from E into an arbitrary locally convex space F is continuous if and only if the restriction of u on any $B \in \mathscr{B}$ is continuous.

Note some elementary assertions concerning $\mathscr{DM}$-spaces:

(a) Any $\mathscr{DB}$-space is simultaneously a $\mathscr{DM}$-space. If X is locally compact, then $(E(X), t_{0c})$ is a $\mathscr{DM}$-space.

(b) The strong dual to a $\mathscr{DM}$-space $(E, \mathscr{B})$ is complete. Expecially, if $\mathscr{B}$ is a countable system, then $(E, \mathscr{B})^*$ is a Fréchet space.

The next theorem will show that for any locally compact X the Λ-structure $(E(X), t_{0c})$ is a special case of a $\mathscr{DM}$-space, namely, it may be represented as an inductive limit of $(E(K), t_c)$, K compact in X (the canonical injections are taken as defining morphisms).

Let $\mathscr{K}$ denote the family of all compact subsets of a locally compact space X. From Corollary 2 of Theorem 6 it follows that for any $K \in \mathscr{K}$ the identical mapping $(E(K), t_c(K)) \to (E(X), t_{0c}(X))$ is continuous, hence the canonical mapping ω of $E = \lim\operatorname{ind}_{K \in \mathscr{K}} (E(K), t_c(K))$ onto $(E(X), t_{0c}(X))$ is also continuous. On the other hand, the continuity of ω^{-1} follows directly from the definition of $(E(X), t_{0c})$ and from the continuity of the embedding $X \to E$. Thus we have proved

Theorem 23. *Let X be a locally compact space, $\mathscr{K}$ the family of all compact subsets in X. Then $(E(X), t_{0c})$ is topologically isomorphic to the inductive limit of all $(E(K), t_c(K))$, $K \in \mathscr{K}$.*

Let $E = (E, \mathscr{B}_1)$ and $F = (F, \mathscr{B}_2)$ be two $\mathscr{DM}$-spaces. A separately continuous bilinear function f on $E \times F$ will be called weakly compact if $u(B)$ is relatively $\sigma(E^*, E^{**})$-compact whenever $B \in \mathscr{B}_2$. Similarly as in Lemma 2 of Section 11 and making use of Theorem 7 of [7] it may be proved that this property is equivalent to the statement that $v(B)$ is relatively $\sigma(F^*, F^{**})$-compact for any $B \in \mathscr{B}_1$. Repeating the same procedure as in Section 11 we may also verify the validity of Lemma 1 for $\mathscr{DM}$-spaces.

The proof of Lemma 3 of Section 11 may be modified for $\mathscr{DM}$-spaces as follows: To prove

$$\langle {}^t u(x_0), y_0 \rangle = \langle x_0, {}^t v(y_0) \rangle$$

for $x_0 \in E^{**}$ and $y_0 \in F^{**}$, we may suppose that y is in the bipolar set B^{00} for some $B \in \mathscr{B}_2$. Hence, for any $\varepsilon > 0$ there exists $x_1 \in E$ with

$$|\langle {}^t u(x_1 - x_0), y \rangle| = |\langle x_1 - x_0, u(y) \rangle| < \tfrac{1}{2}\varepsilon$$

for all $y \in B$ and

$$|\langle x_1 - x_0, {}^t v(y_0) \rangle| < \tfrac{1}{2}\varepsilon .$$

Now we may establish in the same way as in the proof of Lemma 3 the relation

$$|\langle {}^t u(x_0), y_0 \rangle - \langle x_0, {}^t v(y_0) \rangle| < \varepsilon .$$

Hence we have proved

Theorem 24. *Let $(E, \mathscr{B}_1)$ and $(F, \mathscr{B}_2)$ be two $\mathscr{D}\mathscr{M}$-spaces, $f(x, y)$ a weakly continuous bilinear function on $E \times F$. Then the following statements are equivalent:*

(a) *f is weakly continuous.*

(b) *f has a bilinear separately continuous extension to $E^{**} \times F^{**}$ in the weak topologies.*

From Theorem 24 and from Proposition 3 we now obtain

Corollary. *Suppose that f is a separately continuous function on the Cartesian product $X \times Y$ of two locally compact and paracompact spaces. If $\mathscr{K}$ means the system of all compact subsets in Y, u is defined by the usual relation, then the following properties are equivalent:*

(a) *$u(K)$ is weakly relatively compact in $C_0(K)$ for any $K \in \mathscr{K}$.*

(b) *There exists a bilinear and separately continuous extension of f to $C_0^*(X) \times C_0^*(Y)$ in the weak topologies.*

Remark 1. If f is a continuous function on the Cartesian product of two locally compact and paracompact spaces, then we may establish an analogical statement to Theorem 22 for f. Evidently, any such function being uniformly continuous on each $K \times H$, K and H compact in X and Y, it satisfies condition (a) of the last theorem. Hence it has a priori a separately continuous extension to $C_0^*(X) \times C_0^*(Y)$ in the weak topologies.

Remark 2. If X is a locally compact and paracompact space, then we conclude from Theorem 7 of [7] the following criterion for the relative weak compactness in $C_0(X)$:

(a) subset D of $C_0(X)$ is relatively $\sigma(C_0, C_0^*)$-compact if and only if D is bounded in $C_0(X)$ and the canonical bilinear function $(f, x) \to \langle f, x \rangle$ satisfies the double limit condition on each $D \times K$, K compact in X.

References

[1] *N. Bourbaki:* Espaces vectoriels topologiques, Chap. I—V, Hermann, Paris, 1953—1955.

[2] *N. Bourbaki:* Intégration, Chap. I—IV, Hermann, Paris, 1952.

[3] *M. M. Day:* Normed linear spaces, Springer, Berlin—Göttingen—Heidelberg, 1958.

[4] *J. Dieudonné:* Sur un théorème de Šmulian, Archiv der Math., 1952, 436—440.

[5] *A. Grothendieck:* Sur la complétion du dual d'un espace vectoriel localement convexe. C. R. Acad. Sci., Paris, *230*, 1950, 605—606.

[6] *A. Grothendieck:* Sur les applications linéaires faiblement compactes d'espaces du type $C(K)$. Canad. J. Math. *5*, 1953, 129—173.

[7] *A. Grothendieck:* Critères de compacité dans les espaces fonctionnels généraux. Amer. J. Math. *74*, 1952, 168—189.
[8] *A. Grothendieck:* Produits tensoriels topologiques et espaces nucléaires, Memoirs of the Amer. Math. Soc., 1955.
[9] *E. Hewitt:* Rings of real-valued continuous functions. I. TAMS, *2*, 1948, 45—99.
[10] *M. Katětov:* Projectively generated continuity structures: a correction, CMUC *6*, 1965, 251—255.
[11] *G. Köthe:* Topologische lineare Räume I, Berlin, 1960.
[12] *V. Pták:* A combinatorial lemma on the existence of convex means and its application to weak compactness. Amer. Math. Soc., Proc. Symp. in Pure Math. *7*, 1963, 437—450.
[13] *V. Pták:* An Extension Theorem for separately continuous functions and its application to functional analysis. Czech. Math. J. *89*, 1964, 562—581.
[14] *V. Pták:* Weak compactness in convex topological linear spaces, Czech. Math. J. *4*, 1954, 175—186.
[15] *S. Tomášek:* On a theorem of M. Katětov, CMUC 7, 1966, 105—108.
[16] *S. Tomášek:* Über eine Klasse lokalkonvexer Räume, General Topology and its Relations to Modern Analysis and Algebra II, Proc. Symp., Prague 1966, 353—355.
[17] *S. Tomášek:* On a certain class of Λ-structures. I., Czech. Math. J. (to appear).
[18] *S. Tomášek:* Λ-structures and their applications, Thesis, 1966 (in Czech).

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