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ON SPLITTING MIXED ABELIAN GROUPS

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The purpose of this note is to prove two theorems generalizing theorems A1, A2 from [6]. After that, theorems 13–15 from [2] are generalized by using these theorems and some theorems from [7].

By the word "group" we shall always mean an additively written abelian group. A group G is said to be split if its maximal torsion part is a direct summand of G. If H is a subgroup of a torsion free group G then $\{H\}_*^G$ means the pure closure of H in G, i.e. the intersection of all pure subgroups of G containing H. $\hat{\tau}$ denotes the type containing the characteristic τ, $T(G)$ denotes the set of the types of all direct summands J_ι of a completely decomposable group $G = \sum_{\iota \in I d} J_\iota$. In the other cases we adopt the notation used in [1].

Let us note that a torsion free group A is called a K-group if, for every torsion group P, any group G splits whenever G is an extension of the group $H = A \dotplus P$ by a bounded group (see Procházka's paper [3]). In [4] there was proved that any torsion free group of finite rank is a K-group. Finally, let A be a K-group and P an arbitrary torsion group. It is easy to see that if H is a subgroup of $G = A \dotplus P$ such that G/H is bounded, then H splits.

Definition 1. Let H be a subgroup of a group G (mixed in general). We say that H is fully regular in G if the factor-group

$$S/\{S \cap H; T\} \tag{1}$$

is finite for every subgroups $T \subseteq S$ pure in G such that S/T is a torsion free group of finite rank.

Lemma 1. *Let H be a subgroup of a mixed group G such that G/H is a torsion group and P is the maximal torsion part of both groups G and H. Let $P \subseteq H_1 \subseteq$ $\subseteq H_2$ be pure subgroups of H such that H_2/H_1 is of finite rank. Let G_1 and G_2 denote the subgroup of G such that $G_1/P = \{H_1/P\}_*^{G/P}$, $G_2/P = \{H_2/P\}_*^{G/P}$ respectively. Then $G_1 \subseteq G_2$ and G_2/G_1 is of finite rank.*

Proof. Let $g \in G_1$ and $\bar{g} = g + P \in G_1/P$. Then there exists an integer s such that $s\bar{g} \in H_1/P \subseteq H_2/P$. Hence it follows $\bar{g} \in G_2/P$ and $g \in G_2$ so that $G_1 \subseteq G_2$ is proved.

Assume that $r(H_2/H_1) = n - 1$ and let $\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_n$ be arbitrary elements of G_2/G_1. If $g_1, g_2, \ldots, g_n$ are representants of the cosets $\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_n$ then $g_i \in G_2$, $i = 1, 2, \ldots, n$ and from the periodicity of G/H the existence of an integer $m \neq 0$ such that $mg_i \in H_2$, $i = 1, 2, \ldots, n$ follows easily. From $r(H_2/H_1) = n - 1$ it is easy to derive the existence of integers λ_i, $i = 1, 2, \ldots, n$, not all equal to zero, such that $\sum_{i=1}^{n} \lambda_i m g_i \in H_1$. From $H_1 \subseteq G_1$ it follows now $\sum_{i=1}^{n} \alpha\lambda_i m\bar{g}_i = \bar{0}$ (in G_2/G_1) and the elements $\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_n$ are dependent in G_2/G_1 so that $r(G_2/G_1) \leqq n - 1$ and the proof of the lemma is finished.

Theorem 1. *Let G be a mixed group containing a splitting subgroup $H = P \dotplus A$, where P is a torsion group and A a direct sum of torsion free groups of finite rank. If H is fully regular in G then G splits.*

Proof runs on similar principles as the proof of Theorem Al from [6]. Suppose that $A = \sum_{\alpha<\sigma}^{d} A_\alpha$ where $r(A_\alpha) < \infty$ and σ is an arbitrary ordinal. Let T denote the maximal torsion subgroup of G and put $H' = T \dotplus A$ and $H'_\beta = T \dotplus \sum_{\alpha<\beta}^{d} A_\alpha$. Let us define the subgroups G_β of G by the formula $G_\beta/T = \{H'_\beta/T\}_*^{G/T}$. Then G_β is surely pure in G for every $\beta \leqq \sigma$. Finally, it is easy to see that H' is fully regular in G, too.

Using the method of transfinite induction we shall prove that G_β splits for every $\beta \leqq \sigma$, or more precisely that for every $\beta \leqq \sigma$ it is

(2) $$G_\beta = T \dotplus B_\beta \quad \text{and for every} \quad \gamma < \beta \quad \text{it is} \quad B_\gamma \subseteq B_\beta\,.$$

For $\beta = 0$ it is all evident. Firstly, we shall assume that $\beta - 1$ exists. Then by induction hypothesis it holds

(3) $$G_{\beta-1} = T \dotplus B_{\beta-1}\,.$$

Because $A_{\beta-1} \subseteq H'_\beta$ and $G_{\beta-1} \cap H'_\beta = H'_{\beta-1}$, it is true that $G_{\beta-1} \cap A_{\beta-1} = G_{\beta-1} \cap H'_\beta \cap A_{\beta-1} = H'_{\beta-1} \cap A_{\beta-1} = 0$ which implies that the factor-group $G_\beta/B_{\beta-1}$ is an extension of $(G_{\beta-1} \dotplus A_{\beta-1})/B_{\beta-1} = (T \dotplus B_{\beta-1} \dotplus A_{\beta-1})/B_{\beta-1} \cong T \dotplus A_{\beta-1}$ by

$$(G_\beta/B_{\beta-1})/((G_{\beta-1} \dotplus A_{\beta-1})/B_{\beta-1}) \cong G_\beta/(G_{\beta-1} \dotplus A_{\beta-1}) = G_\beta/\{G_{\beta-1}, G_\beta \cap H'\}\,.$$

By Lemma 1, the factor-group $G_\beta/G_{\beta-1}$ is of finite rank, so that by Definition 1 and by hypothesis the factor-group $G_\beta/(G_{\beta-1} \dotplus A_{\beta-1})$ is finite.

The group $A_{\beta-1}$ as a rank finite group is a K-group (see e.g. Procházka's papers [3], [4]) so that $G_\beta/B_{\beta-1}$ splits,

(4) $$G_\beta/B_{\beta-1} = B_\beta/B_{\beta-1} \dotplus G_{\beta-1}/B_{\beta-1}$$

where $G_{\beta-1}/B_{\beta-1}$ is the maximal torsion subgroup of $G_\beta/B_{\beta-1}$. In fact, $G_{\beta-1}/B_{\beta-1}$ is a torsion group by (3) and it is maximal because $G_\beta/B_{\beta-1}/G_{\beta-1}/B_{\beta-1} \cong G_\beta/G_{\beta-1}$ is torsion free by Lemma 1. Clearly, $T \cap B_\beta = 0$, $B_{\beta-1} \subseteq B_\beta$. From (4) and (3) it may be easily derived that (2) is true.

Secondly, let β be a limit ordinal. Then clearly $G_\beta = \bigcup_{\gamma<\beta} G_\gamma$ and by induction hypothesis $G_\gamma = T \dotplus B_\gamma$ for all $\gamma < \beta$ and $B_\delta \subseteq B_\gamma$ for all $\delta < \gamma < \beta$ so that we can put $B_\beta = \bigcup_{\gamma<\beta} B_\gamma$. For an arbitrary $g \in G_\beta$ there exists $\gamma < \beta$ such that $g = t + b$, $t \in T$, $b \in B_\gamma \subseteq B_\beta$, i.e. $g \in T + B_\beta$. From this fact the splittingness of G_β easily follows.

In particular, for $\beta = \sigma$ it is $G = G_\sigma = T \dotplus B_\sigma$ so that the proof of Theorem 1 is finished.

Theorem 2. *Let $G = T \dotplus B$ be a splitting mixed group where T is a torsion group and B torsion free and H is a subgroup of G with the maximal torsion subgroup P. If either*

1) T/P is bounded and B is of finite rank,

or

2) $B = \sum_{\lambda \in \Lambda}^{d} B_\lambda$ is a direct sum of K-groups and for every $\lambda \in \Lambda$ the factor-group $B_\lambda/B_\lambda \cap H$ is bounded,

then H splits, too.

Proof. Firstly, let T/P be bounded and B be of finite rank. Put $K = \{T, H\}$ so that $K = T \dotplus K_1$, where $K_1 = K \cap B$. Further, $K/H = \{T, H\}\, H \cong T/T \cap H = T/P$ is bounded. K_1 as a subgroup of B is of finite rank, i.e. it is a K-group and thus H splits.

Secondly, we can assume that Λ is the set of ordinals $\alpha < \sigma$. Put

(5) $$G_\beta = T \dotplus \sum_{\alpha<\beta}^{d} B_\alpha$$

and

(6) $$H_\beta = G_\beta \cap H$$

for every ordinal $\beta \leqq \sigma$. Clearly, G_β is a pure subgroup of G for every $\beta \leqq \sigma$. Using the method of transfinite induction we shall prove that for every $\beta \leqq \sigma$ it is

(7) $$H_\beta = P \dotplus A_\beta \quad \text{and for} \quad \gamma < \beta \quad \text{it is} \quad \alpha_\gamma \subseteq A_\beta\,.$$

For $\beta = 0$ it is all evident. Firstly, we shall assume that $\beta - 1$ exists. Then by induction hypothesis it holds

(8) $$H_{\beta-1} = P \dotplus A_{\beta-1}\,.$$

By hypothesis and by (6) the factor-group $G_{\beta-1}/H_{\beta-1}$ is periodical so that to an arbitrary $g \in G_{\beta-1}$ there exists an integer $n \neq 0$ (depending on g) such that $ng \in H_{\beta-1}$. By (8) it is $ng = p + a$ where $p \in P$, $a \in A_{\beta-1}$. From the periodicity of P the existence of a non-zero integer m follows such that $mp = 0$. Altogether we have $mng = ma \in \in A_{\beta-1}$ so that the factor-group

$$(9) \qquad G_{\beta-1}/A_{\beta-1}$$

is a torsion group. Further, by (5) it is $G_\beta = G_{\beta-1} \dotplus B_{\beta-1}$. From $A_{\beta-1} \cap B_{\beta-1} \subseteq \subseteq H_{\beta-1} \cap B_{\beta-1} \subseteq G_{\beta-1} \cap B_{\beta-1} = 0$ it easily follows

$$(10) \qquad G_\beta/A_{\beta-1} = G_{\beta-1}/A_{\beta-1} \dotplus (B_{\beta-1} \dotplus A_{\beta-1})/A_{\beta-1}\,.$$

Due to the isomorphism

$$(11) \qquad (B_{\beta-1} \dotplus A_{\beta-1})/A_{\beta-1} \cong B_{\beta-1}$$

the factor-group $G_\beta/A_{\beta-1}$ splits by hypothesis and by (9) Put $K = \{H_\beta, B_{\beta-1}\}$. Then $B_{\beta-1} \dotplus A_{\beta-1} \subseteq K$ and

$$(12) \qquad K/A_{\beta-1} = (G_{\beta-1}/A_{\beta-1} \cap K/A_{\beta-1}) \dotplus (B_{\beta-1} \dotplus A_{\beta-1})/A_{\beta-1}\,.$$

Hence the factor-group $K/A_{\beta-1}$ splits by (11), (9) and its torsion free direct summand is a K-group by hypothesis. Further, $H_\beta/A_{\beta-1} \subseteq K/A_{\beta-1}$ and the factor-group $(K/A_{\beta-1})/(H_\beta/A_{\beta-1}) \cong K/H_\beta = \{H_\beta, B_{\beta-1}\}/H_\beta \cong B_{\beta-1}/B_{\beta-1} \cap H_\beta = B_{\beta-1}/B_{\beta-1} \cap H$ is bounded by hypothesis so that $H_\beta/A_{\beta-1}$ splits by the definition of a K-group. The maximal torsion subgroup of $H_\beta/A_{\beta-1}$ is $H_{\beta-1}/A_{\beta-1}$. In fact, $H_{\beta-1}/A_{\beta-1}$ is a torsion group by (8) and $(H_\beta/A_{\beta-1})/(H_{\beta-1}/A_{\beta-1})$ is torsion free because $(H_\beta/A_{\beta-1})/(H_{\beta-1}/A_{\beta-1}) \cong H_\beta/H_{\beta-1} = H_\beta/G_{\beta-1} \cap H_\beta \cong \{G_{\beta-1}, H_\beta\}/G_{\beta-1} \subseteq \subseteq G_\beta/G_{\beta-1} \cong B_{\beta-1}$. Then we can write

$$(13) \qquad H_\beta/A_{\beta-1} = H_{\beta-1}/A_{\beta-1} \dotplus A_\beta/A_{\beta-1}$$

where $A_\beta/A_{\beta-1}$ is a suitable torsion free subgroup of $H_\beta/A_{\beta-1}$.

Clearly, $A_\beta \cap P = 0$. If $h \in H_\beta$ is an arbitrary element, then $h + A_{\beta-1} = (a + + A_{\beta-1}) + (h' + A_{\beta-1})$, $a \in A_\beta$, $h' \in H_{\beta-1}$, so that (7) now easily follows in view of (8).

Secondly, let β be a limit ordinal. It is easy to see that $H_\beta = \bigcup_{\gamma<\beta} H_\gamma$ and by induction hypothesis $H_\gamma = P \dotplus A_\gamma$ for all $\gamma < \beta$ and $A_\delta \subseteq A_\gamma$ for all $\delta < \gamma < \beta$. Put $A_\beta = = \bigcup_{\gamma<\beta} A_\gamma$. For an arbitrary $h \in H_\beta$ there exists $\gamma < \beta$ such that $g = p + a$, $p \in P$, $a \in A_\gamma \subseteq A_\beta$, i.e. $h \in P \dotplus A_\beta$. From this fact the splittingness of H_β easily follows.

In particular, for $\beta = \sigma$ it is $H = H_\sigma = P \dotplus A_\sigma$ and the proof is now finished.

Definition 2. Let H be a subgroup of the group G. We say that H is strongly regular in G if the factor-group $S/S \cap H$ is finite for every torsion free subgroup S of finite rank pure in G.

Theorem 3. *Let G be a mixed group with the maximal torsion subgroup T containing a splitting subgroup H of the form $H = P \dotplus A$ where P is a torsion and A a direct sum of countably many rank finite groups. If $\{H, T\}/T$ is strongly regular in G/T then G splits.*

Proof. If A is of finite rank then $G(T \dotplus A) \cong (G/T)/((T \dotplus A)/T) = (G/T)/(\{H, T\}/T)$ is finite by hypothesis, and G splits by Theorem 3 from [5].

Let us suppose that $A = \sum_{n=1}^{\infty}{}_d A_n$, $r(A_n) < \infty$, $n = 1, 2, \ldots$ Put $H' = T \dotplus A$, $H = T + \sum_{i<n}{}_d A_i$ and let G_n be a pure subgroup of G defined by the formula $G_n/T = \{H'_n/T\}_*^{G/T}$. Now we shall proceed by induction by n. Firstly, $G_1 = T$ splits. If $G_{n-1} = T \dotplus B_{n-1}$ splits then for $K = G_{n-1} \dotplus A_{n-1}$, $(G_n/B_{n-1})/(K/B_{n-1}) \cong G_n/K \cong (G_n/T)/(K/T)$ is a finite group as a homomorphic image of $(G_n/T)/(H'_n/T)$. Then G_n/B_{n-1} splits by Theorem 3 from [5]. It is easy to see that

$$G_n/B_{n-1} = G_{n-1}/B_{n-1} \dotplus B_n/B_{n-1} \tag{14}$$

for a suitable subgroup $B_n \subseteq G_n$. Now the proof proceeds along the same lines as in Theorem 1 (among the limit ordinals only ω must be discussed).

Definition 3. We say that the subgroup H of the group G is regular in G, if the factor-group $S/S \cap H$ is finite for every torsion free rank one subgroup S pure in G.

Note that Baer introduced the following classes of torsion free groups (see e.g. [1], d. 174). Define Γ_1 as the set of all countable torsion free groups. If α is an ordinal, $\alpha > 1$, then we let the torsion free group G belong to Γ_α if $G \notin \Gamma_\beta$ for $\beta < \alpha$ and there exists a pure subgroup S of finite rank of G such that G/S is a direct sum of groups belonging to classes of indices less than α.

Now we shall formulate three theorems (without proofs) which were stated in [7].

Theorem A (see Theorem 4 from [7]): *Let G be a torsion free group containing a completely decomposable homogeneous subgroup H such that G/H is a torsion group. Then $G \cong H$ if and only if*

1) *$G \in \Gamma_\alpha$ for some ordinal α,*

2) *H is strongly regular in G.*

Theorem B (see Theorem 1 from [7]). *Let G be a torsion free group containing a completely decomposable subgroup H such that*

1) *$T(H)$ satisfies the maximum condition,*

2) *for any two incomparable types $\hat{\tau}_1, \hat{\tau}_2$ from $T(H)$ it is $\hat{\tau}_1 \vee \hat{\tau}_2 = \hat{R}$.*[1]) *If H is fully regular in G then $G \cong H$.*

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[1]) $\hat{R}$ denotes the greatest element of the lattice of all types.

Theorem C (see Theorem 2 from [7]). *Let G be a completely decomposable torsion free group such that $T(G)$ satisfies conditions* 1) *and* 2) *stated in Theorem B. If H is regular in G then $G \cong H$.*

Now we are ready to prove several theorems, some of which are generalizations of the theorems 13–15 from [2]. This fact we shall not prove here, because it can be easily derived from some theorems and corollaries proved in [7].

Theorem 4. *Let G be a mixed group with the maximal torsion subgroup T containing a splitting subgroup H of the form $H = P \dotplus A$, where P is a torsion group and A a torsion free completely decomposable group such that $T(A)$ satisfies conditions* 1) *and* 2) *from Theorem B. If H is fully regular in G then G splits, $G = T \dotplus A_0$ and $A_0 \cong A$.*

Proof. G splits by Theorem 1, $G = T \dotplus A_0$. Further, $H \subseteq H_0 = T \dotplus A \subseteq G$ and hence $H_0 = T \dotplus A_0 \cap H_0$. Let $U \subseteq S$ be pure subgroups of A_0 such that S/U is a torsion free group of finite rank. From the purity of A_0 in G it follows by Definition 1 that the factor-group $S/\{S \cap H, U\} = S/\{S \cap (A_0 \cap H), U\}$ is finite. The inclusion $H \subseteq H_0$ shows that $A_0 \cap H_0$ is fully regular in A_0. As $A_0 \cap H_0 \cong H_0/T \cong$ $\cong A$ fulfils all the conditions of Theorem B, the isomorphism $A_0 \cong A_0 \cap H_0$ completes the proof.

Theorem 5. *Let G be a splitting group, $G = T \dotplus A_0$ where T is a torsion group and A_0 a completely decomposable torsion free group such that $T(A_0)$ satisfies conditions* 1) *and* 2) *from Theorem B. If H is a regular subgroup of G then H splits, $H = P \dotplus A$ and $A \cong A_0$.*

Proof. By Theorem 2 H splits, $H = P \dotplus A$. As in the preceding proof it is $H \subseteq$ $\subseteq H_0 = T \dotplus A = T \dotplus (A_0 \cap H_0)$ so that $A \cong A_0 \cap H_0$. It is not too difficult to show that $A_0 \cap H_0$ is regular in A_0, hence Theorem C completes the proof.

Theorem 6. *Let G be a mixed group with the maximal torsion subgroup T containing a splitting subgroup H of the form $H = A \dotplus P$ where P is a torsion group and A a homogeneous completely decomposable torsion free group. If G/T is countable and $\{H, T\}/T$ strongly regular in G/T then G splits, $G = A_0 \dotplus T$ and $A_0 \cong A$.*

Proof. Let us denote $H_0 = \{H, T\} = T \dotplus A \subseteq G$. Then $A \cong H_0/T \subseteq G/T$ is a direct sum of countably many rank one groups and H_0/T is clearly strongly regular in G/T. By Theorem 3 G splits, $G = T \dotplus A_0$. Now G/T is a torsion free countable group containing $H_0/T \cong A$ as a subgroup, so that by Theorem A (for $\alpha = 1$) it is $G/T \cong H_0/T$ and the theorem easily follows.

Theorem 7. *Let G be a mixed group with the maximal torsion subgroup T containing a splitting subgroup H of the form $H = A \dotplus P$ where P is a torsion group and A a homogeneous completely decomposable torsion free group. If G contains* *a subgroup G_1 such that $H \subseteq G_1 \subseteq G$, H is fully regular in G_1, $\{G_1, T\}/T$ is strongly regular in G/T and G/G_1 is countable, then G splits, $G = A_0 \dotplus T$ and $A_0 \cong A$.*

Proof. By Theorem 4 G_1 splits, $G_1 = Q \dotplus A_1$ and $A_1 \cong A$. If $g \in G \dot{-} G_1$ is an arbitrary element then by hypothesis it is $r\{g + T\}_*^{G/T} \subseteq \{G_1, T\}/T$ for a suitable non-zero integer r, i.e. $rg = a + t$, $a \in A_1$, $t \in T$. If s is the order of t then for $m = rs$ it is $mg \in A_1$, i.e. mg has a non-zero component in finitely many direct summands of a given complete decomposition of $A_1 = \sum_{\iota \in I}^d J_\iota$. Let us choose one element in each coset of G/G_1 and let us denote by M the set of all these elements. If we denote by I_1 the set of all indices $\iota \in I$ such that J_ι contains a non-zero component of at least one element mg, $g \in M$[2]) (m depending on g), then I_1 is clearly countable (because M is countable). Put $I_2 = I \dot{-} I_1$, $G' = \{T \dotplus \sum_{\iota \in I_1}^d J_\iota; M\}$, $G'' = \sum_{\iota \in I_2}^d J_\iota$. It is $G' \cap G'' = 0$, because for $g \in G' \cap G''$ it is $mg \in (\sum_{\iota \in I_1}^d J_\iota) \cap G'' = 0$ for a suitable integer m and hence the torsion free character of G'' implies $g = 0$. On the other hand $G = \{G_1, M\} = \{G', G''\}$ so that $G = G' \dotplus G''$.

Further, G'/T is countable because the elements from $\{\sum_{\iota \in I_1}^d J_\iota, M\}$ form the set of representatives of the cosets of G'/T. If we denote $G_1' = T \dotplus \sum_{\iota \in I_1}^d J_\iota$, then clearly $G_1' = G' \cap G_1$ and from Definition 2 now easily follows that G_1'/T is strongly regular in G'/T. By using Theorem 6 our assertion now follows without complications.

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[2]) The set of those J_ι, $\iota \in I$ in which mg has a non-zero component does not depend on the choice of the integer m for which $mg \in A_1$. Surely, if t is the least positive integer for which $tg \in A_1$, then $m = tq + r$, $0 \leqq r < t$. For $r \neq 0$ it is $rg = mg - qtg \in A_1$ a contradiction and the assertion follows.