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CENTRALLY SYMMETRIC HASSE DIAGRAMS
OF FINITE MODULAR LATTICES

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In [3] a centrally symmetric graph, or S -graph, is defined as an undirected graph without loops and multiple edges fulfilling the following conditions:

- (1) G contains at least one edge;
- (2) for each triplet $\{x, y, z\}$ of its vertices such that $\varrho_G(y, z) = 1$ we have $\varrho_G(x, y) \neq \varrho_G(x, z)$;
- (3) for each vertex x of G exactly one vertex \bar{x} exists such that for each vertex w of a neighbourhood of \bar{x} we have $\varrho_G(x, \bar{x}) > \varrho_G(x, w)$.

Here $\varrho_G(a, b)$ denotes the distance of a and b in G . The vertices x and \bar{x} are called opposite to each other.

In [3] the following theorems are proved.

(A) *If for each chosen vertex x of G there exists a Jordan-Dedekind lattice such that its Hasse diagram (see [2], [4]) is isomorphic to G and its greatest element is x , then G is an S -graph.*

(B) *If G is an arbitrary S -graph and x is its vertex, then $\bar{\bar{x}} = x$.*

(C) *If G is an arbitrary S -graph and d is its diameter, then arbitrary two opposite vertices and only such two vertices have the distance d .*

Further in [3] A. KOTZIG suggests to study such S -graphs which satisfy the assumption of (A). He conjectures that these graphs are C^1, K_6, K_8, \dots and Cartesian products of these graphs. This conjecture is expressed also in [1], among the unsolved problems. The symbol C^1 denotes the graph consisting of exactly one edge and its end vertices, the symbol K_n denotes the circuit with n vertices.

In this paper we shall study only S -graphs which satisfy the assumption of (A) so that the corresponding lattices are modular and finite.

Theorem. *Let L be a finite modular lattice with n atoms such that its Hasse diagram*

is an S -graph. Then L is a Boolean algebra and its Hasse diagram is the graph of the n -dimensional cube.

Remark. The assumption of this theorem is more general than that of (A). On the other side, it is evident that the graph of an n -dimensional cube, because of its high degree of symmetry, satisfies not only the assumption of this theorem, but even the assumption of (A).

This result does not contradict to Kotzig's conjecture, because the graph of the n -dimensional cube is the n -th Cartesian power of the graph C^1 .

Before proving Theorem we shall state some lemmas.

By $d(x)$ the dimension function on L is denoted.

Lemma 1. *Let L be a finite modular lattice whose Hasse diagram is an S -graph. Then for each $a \in L$ we have $a \wedge \bar{a} = O$, $a \vee \bar{a} = I$.*

Remark. We do not distinguish the elements of L and the vertices of the Hasse diagram of L .

Proof. Assume that $a \wedge \bar{a} = b \succ O$. Then there exists a saturated chain C_1 of the length $d(a) - d(b)$ in L whose least element is b and greatest element is a and a saturated chain C_2 of the length $d(\bar{a}) - d(b)$ in L whose least element is b and greatest element is \bar{a} . In the Hasse diagram of L two elementary paths of the same lengths correspond to these chains. The union of these paths is a path joining a and \bar{a} of the length $l = d(a) + d(\bar{a}) - 2d(b)$. As L is modular, we have $d(a) + d(\bar{a}) = d(a \wedge \bar{a}) + d(a \vee \bar{a}) = d(b) + d(a \vee \bar{a})$, so $l = d(a \vee \bar{a}) - d(b)$. As $a \vee \bar{a} \leq I$, we have $d(a \vee \bar{a}) \leq d(I) = d(L)$, and as $b \succ O$, we have $d(b) > d(O) = 0$. This implies $l < d(L)$ which is a contradiction because the diameter of the Hasse diagram of L is evidently $d(L)$. So we have proved $a \wedge \bar{a} = O$. The proof of $a \vee \bar{a} = I$ is dual.

Lemma 2. *Let G be the Hasse diagram of a finite modular lattice L . Let a, b be two of its vertices (and at the same time elements of L) and let P_0 be an elementary path of the minimal length l joining $a \in L$ and $b \in L$. Then there exists a path P' of the length l joining a and b so that $P' = P'_1 \cup P'_2$ where P'_1 and P'_2 are Hasse diagrams of two chains C_1 and C_2 in L , the least element of C_1 or C_2 is a or b respectively; the chains C_1 and C_2 have a common greatest element which is their only common element.*

Proof. As L is a modular lattice, there exists a dimension function $d(x)$ on L such that $d(x) + d(y) = d(x \wedge y) + d(x \vee y)$ for arbitrary x and y of L . If two elements x and y of L are joined by an edge in G , then either $d(y) = d(x) + 1$ or $d(y) = d(x) - 1$. If P is an elementary path in the Hasse diagram of L , we denote by $D(P)$ the sum of $d(x)$ for all vertices x of the path P . Now let P_0 be an elementary path of the length l joining a and b in the Hasse diagram of L . Let P_0 contain three elements x, y, z such that xy and yz are the edges of P_0 and $d(x) = d(z)$, $d(y) = d(x) - 1$. Then x and z

cover y , so $y = x \wedge z$ and, as L is modular, $x \vee z$ covers x and z . Denote $x \vee z = t$. Omit the vertex y and the edges xy and yz from P_0 and substitute them by the vertex t and the edges xt and tz . We obtain a path P_1 again of the length l joining a and b . We have $D(P_1) = D(P_0) + 2$. We continue this process and obtain a sequence P_0, P_1, P_2, \dots of the paths between a and b having all the same length l such that $D(P_i)$ increases. As $D(P_i)$ increases, no path can occur in the sequence more than once. As L is finite, such a sequence can have only a finite number of elements. Thus the last path P' in the sequence is a path in which no element (vertex) is covered by two other vertices of the path. Such a path must be a path described in the assertion of the lemma. As all paths of the sequence have the length l , also P' has this length.

Lemma 3. *Let L be a finite modular lattice whose Hasse diagram is an S -graph. If $a \in L, b \in L, b \neq \bar{a}$, then either $a \vee b < I$ or $a \wedge b > O$.*

Proof. As $b \neq \bar{a}$, the distance between a and b in the Hasse diagram of L is less than $d(L)$. Let P be a shortest elementary path between a and b , let l be its length. According to Lemma 2 there exists a path P' of the length l between a and b such that $P' = P'_1 \cup P'_2$ where P'_1 and P'_2 are Hasse diagrams of two saturated chains C_1 and C_2 in L , the least element of C_1 or C_2 is a or b respectively, the chains C_1 and C_2 have a common greatest element which is their only common element. Let this element be denoted by c . The length of C_1 is $d(c) - d(a)$, the length of C_2 is $d(c) - d(b)$. Thus the length of P' (and also of P) is $l = 2d(c) - d(a) - d(b)$. As $l < d(L)$, we have $2d(c) - d(a) - d(b) < d(L)$. Assume that $a \wedge b = O$. Then $d(a) + d(b) = d(a \wedge b) + d(a \vee b) = d(O) + d(a \vee b) = d(a \vee b)$. So we have $2d(c) < d(L) + d(a \vee b)$. The element c is greater than both a and b , so $c \geq a \vee b$ and $d(c) \geq d(a \vee b)$. Thus $2d(a \vee b) \leq 2d(c) < d(L) + d(a \vee b)$, from which $d(a \vee b) < d(L)$ follows and, as $d(L) = d(I)$, we have $a \vee b < I$. So $a \wedge b = O$ implies $a \vee b < I$, q.e.d.

Proof of Theorem. If $a \in L$, then according to Lemma 1 there exists at least one complement of a , the opposite element \bar{a} , and according to Lemma 3 no other complement of a exists. So L is a uniquely complementary modular lattice. According to [4], p. 125, the lattice L is distributive. As L is distributive and uniquely complementary, it is a Boolean algebra. As it is well-known, the Hasse diagram of the finite Boolean algebra with n atoms is the graph of the n -dimensional cube.

References

- [1] Beiträge zur Graphentheorie. Leipzig 1968.
- [2] G. Birkhoff: Lattice Theory. New York 1948.
- [3] A. Kotzig: О центрально симметрических графах. Czech. Math. J. 18 (1968), 606—615.
- [4] G. Szász: Einführung in die Verbandstheorie. Leipzig 1962.

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