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THE EXISTENCE OF OSCILLATORY SOLUTIONS FOR A NONLINEAR ODD ORDER DIFFERENTIAL EQUATION

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The equation to be considered is

$$y^{(n)} + q(t)\, y^{\gamma} = 0 \tag{1}$$

where n is an odd integer $\geqq 3$, $q(t) \geqq 0$ and continuous on some half line $[a, \infty)$, and γ is the quotient of odd, positive integers. This equation has been considered recently by KIGURADZE [2], [3], [4] and LIČKO and ŠVEC [5]. The case $\gamma = 1$ has been considered in [2]; the case $n = 3$ has been considered in [1] and [6].

We begin with some definitions and lemmas. A solution $y(t)$ of (1) is called oscillatory if it does not have a last zero, i.e., if $y(t) = 0$ for some t, then there is a $\bar{t} > t$ such that $y(\bar{t}) = 0$. Otherwise the solution is called nonoscillatory. A solution is called extendable if it exists on some half line $[b, \infty)$. A nontrivial solution of (1) is called singular ([4]) if it is identically zero on some half line $[c, \infty)$. A solution can only be singular if $0 < \gamma < 1$.

Lemma 1. *If $0 < \gamma < 1$, then all solutions of (1) are extendable. If $1 < \gamma$, all nonextendable solutions are oscillatory.*

These two facts have been proven for the case $n = 3$ in [1]. The generalization to arbitrary n is not difficult. The following lemma shows when an oscillatory solution is not singular.

Lemma 2. *If $y(t)$ is an oscillatory solution of (1) and there is a point t_0 such that $y(t_0) = 0$, $y^{(i)}(t_0) > 0$ for $i = 1, \ldots, n - 1$, then $y(t)$ is not singular.*

Proof. Since $y(t)$ is oscillatory there is a $t_1 > t_0$ such that $y(t_1) = 0$ and $y(t) > 0$ in (t_0, t_1). Since $y^{(n)}(t) \leqq 0$ in $[t_0, t_1]$, each derivative $y^{(i)}$, $i = 1, \ldots, n - 1$ will become negative and stay negative as t varies from t_0 to t_1. Since, as t varies from t_0 to t_1, the derivatives become negative in descending order of (i), there are points

r_1, r_2 with $t_0 < r_1 < r_2 < t_1$ such that $y^{(n-1)}(r_1) = 0$, $y^{(n-3)}(r_2) = 0$ and such that $y^{(n-3)}(t) > 0$ on $[t_0, r_2)$ while $y^{(n-1)}(t) < 0$ on $(r_1, t_1]$. Therefore $y^{(n-2)}(r_1) > > y^{(n-2)}(t)$ for $t_0 \leqq t \leqq r_1$.

Now multiply (1) by $y^{(n-3)}(t)$ and integrate between r_1 and r_2 to obtain

$$y^{(n-1)}(s)\, y^{(n-3)}(s)\Big|_{r_1}^{r_2} - \left(\frac{y^{(n-2)}(s)}{2}\right)^2\Big|_{r_1}^{r_2} + \int_{r_1}^{r_2} q(s)\,(y(s))^\gamma\,(y^{(n-3)}(s))\,ds = 0\,.$$

Therefore

$$\left(\frac{y^{(n-2)}(r_2)}{2}\right)^2 - \left(\frac{y^{(n-2)}(r_1)}{2}\right)^2 = \int_{r_1}^{r_2} q(s)\,(y(s))^\gamma\,(y(s))^{(n-3)}\,ds \geqq 0$$

and so

$$|y^{(n-2)}(t_1)| > |y^{(n-2)}(r_2)| \geqq |y^{(n-2)}(r_1)| - |y^{(n-2)}(t_0)|\,.$$

Let $\{t_n\}$ be the increasing sequence of zeros of $y(t)$. It then follows from the above argument (since $-y(t)$ is also a solution of (1)) that $|y^{(n-2)}(t_n)|$ is increasing in n. Therefore $y(t)$ is not singular.

Lemma 3. *Suppose that $q(t) > 0$ and $q'(t) \leqq 0$, for $t \geqq a$. If $y(t)$ is an oscillatory solution satisfying the same hypothesis as in Lemma 2, then $|y(t)|$ is nondecreasing at its relative maxima and minima after t_0.*

Proof. Let u_1 be the maxima of $y(t)$ between t_0 and t_1, and u_3 the minima of $y(t)$ between t_1 and t_2 where t_1 and t_2 are as given in the proof of Lemma 2. Let u_2 be the first zero of $y^{(n-1)}(t)$ after u_1. Thus we have $u_1 < u_2 < u_3$. Now multiply (1) by $y'(t)/q(t)$ and integrate between u_1 and u_2 to obtain

$$\frac{y^{(n-1)}(s)\, y'(s)}{q(s)}\Big|_{u_1}^{u_2} - \int_{u_1}^{u_2} \frac{y^{(n-1)}(s)\, y''(s)}{q(s)}\,ds +$$
$$+ \int_{u_1}^{u_2} \frac{y^{(n-1)}(s)\, y'(s)\, q'(s)\,ds}{(q(s))^2} + \frac{(y(s))^{1+\gamma}}{1+\gamma}\Big|_{u_1}^{u_2} = 0\,.$$

Since $y^{n-1}(t) \leqq 0$, $y''(t) \leqq 0$, and $y'(t) \leqq 0$ on $[u_1, u_2]$, we see that $|y(u_3)| > |y(u_2)| > > |y(u_1)|$. By repeated application of this argument, the lemma follows.

The proof of the existence of oscillatory solutions depends on the behavior of nonoscillatory solutions. The next three lemmas are concerned with this. The next lemma is merely a restatement of a lemma of Kiguradze [2], and will not be proved here.

Lemma 4. *If $f(t)$ is a real valued function defined on $[t_0, \infty)$ for some t_0 such that $f(t), \ldots, f^{(n)}(t) > 0$ on $[t_0, \infty)$ for $n \geqq 1$ and $f^{(n+1)}(t) \leqq 0$ on $[t_0, \infty)$, then there is* *a number* $K > 0$ *and a* $t_1 > t_0$ *such that*

$$\frac{f^{(i)}(t)}{f^{(i+1)}(t)} \geqq Kt \quad i = 1, \dots, n-1$$

for $t \geqq t_1$.

Lemma 5. *Let* $f(t)$ *be a positive real valued function defined on* $[t_0, \infty)$ *for some* t_0 *and satisfying* $(-1)^{i+1} f^{(i)}(t) > 0$ *for* $i = 1, \dots, n$ *and* $f^{(n+1)}(t) \leqq 0$ *on* $[t_0, \infty)$. *Then there is a number* $L > 0$ *and a* $t_1 \geqq t_0$ *such that*

$$\frac{f(t)}{f^{(n)}(t)} \geqq Lt^n$$

for $t \geqq t_1$.

Proof. Consider the function $G(t)$ defined by

$$G(t) = f(t)(t - t_0) - f'(t)\frac{(t - t_0)^2}{2} + \dots$$

$$\dots + \frac{(-1)^i f^{(i)}(t)(t - t_0)^{i+1}}{(i+1)!} + \dots + \frac{f^{(n)}(t)(t - t_0)^{n+1}}{(n+1)!}.$$

Then $G(t_0) = 0$ and

$$G'(t) = f(t) - \frac{f^{(n+1)}(t)(t - t_0)^{n+1}}{(n+1)!} > 0$$

for $t > t_0$. Therefore $G(t) > 0$ for $t > t_0$ which means that

$$f(t)(t - t_0) - \frac{f^{(n)}(t)(t - t_0)^{n+1}}{(n+1)!} \geqq 0$$

for $t \geqq t_0$. Therefore

$$\frac{f(t)}{f^{(n)}(t)} \geqq \frac{(t - t_0)^n}{(n+1)!} \geqq Lt^n$$

for $t \geqq t_1 \equiv 2t_0$ and $L = 1/(n+1)!\,2^n$.

Lemma 6. *If* $y(t)$ *is a nonoscillatory solution of* (1) *and*

a) $\int^\infty s^{n-2+\gamma} q(s)\,ds = \infty$ *if* $1 < \gamma$

b) $\int^\infty s^{(n-1)\gamma} q(s)\,ds = \infty$ *if* $0 < \gamma < 1$

then $|y(t)|$ *is eventually nonincreasing.*

Proof. If $1 < \gamma$ this lemma has been proved by Kiguradze [3]. However it can also be proved in the same manner as we will prove it for $0 < \gamma < 1$.

Suppose then that $0 < \gamma < 1$ and that $y(t)$ is a nonoscillatory solution. By Lemma 1, $y(t)$ is extendable and we can suppose that $y(t) > 0$ for $t \geqq t_0$ (since $-y(t)$ is also a solution). To obtain a contradiction suppose that $y'(t) > 0$ for $t \geqq t_1 \geqq t_0$. Then $y^{(n)}(t) < 0$ for $t \geqq t_1$ and thus there is a $t_2 \geqq t_1$ and a k, $1 \leqq k \leqq n-2$, such that $(-1)^i\, y^{(n-i)}(t) < 0$ for $i = 1, \ldots, k$ and $t \geqq t_2$ and $y^{(n-i)}(t) > 0$ for $i = k+1, \ldots, n$ and $t \geqq t_2$. Therefore by Lemmas 4 and 5 we see that there is a $K > 0$ and a $t_3 \geqq t_2$ such that

$$\frac{y(t)}{y^{(n-1)}(t)} \geqq K t^{n-1}$$

for $t \geqq t_3$.

We now multiply (1) by $1/(y^{(n-1)}(t))^\gamma$ and integrate between t_3 and t, $t_3 \leqq t$ to obtain

$$\left.\frac{y^{(n-1)}(s)^{1-\gamma}}{1-\gamma}\right|_{t_1}^{t} \leqq -K \int_{t_1}^{t} s^{(n-1)\gamma}\, q(s)\, \mathrm{d}s$$

but this is a contradiction since the right hand side $\to -\infty$ as $t \to \infty$ and the left hand side is bounded as $t \to \infty$.

We now are ready to prove the major result of this paper. The case $n = 3$ has already been discussed in [1]. See also [4].

Theorem. *Suppose that*

a) $\int^\infty s^{n-2+\gamma}\, q(s)\, \mathrm{d}s = \infty$ *if* $1 < \gamma$,

b) $\int^\infty s^{(n-1)\gamma}\, q(s)\, \mathrm{d}s = \infty$ *if* $0 < \gamma < 1$.

Then (1) *has a nonsingular oscillatory solution* $y_1(t)$. *If* $q(t) > 0$ *and* $q'(t) \leqq 0$, *then* $|y_1|$ *is increasing at its successive maxima and minima.*

Proof. If $y(t)$ is any solution of (1), we define the functional $F(y(t))$ by

$$F(y(t)) = -y^{(n-1)}(t)\, y(t) + y^{(n-2)}(t)\, y'(t) + \ldots + (-1)^{(n-1)/2} (y^{(n-1)/2}(t))^2 .$$

Then it is easily verified that

$$F(y(t)) = F(y(t_0)) + \int_{t_0}^{t} q(s)\, (y(s))^{\gamma+1}\, \mathrm{d}s$$

for $t \geqq t_0$. Therefore $F(y(t))$ is increasing in t along every solution of (1). Now suppose that $y(t)$ is a nonoscillatory solution of (1) and suppose that $y(t) > 0$ for $t \geqq t_0$. Suppose there is a $t_1 \geqq t_0$ such that $F(y(t_1)) > 0$. It is asserted that $y'(t) \geqq 0$ eventually. Suppose $y'(t) \leqq 0$ eventually. Then $\lim_{t\to\infty} y^{(i)}(t) = 0$, $i = 1, \ldots, n-1$ and therefore $\lim_{t\to\infty} F(y(t)) = 0$. This contradicts $F(y(t_1)) > 0$ and $F(y(t))$ increasing in t.

We define the solution $y_1(t)$ by the conditions $y_1(t_2) = 0$, $y_1^{(i)}(t_2) > 0$, $i = 1, \dots$ $\dots, n - 1$, and $F(y(t_2)) > 0$. Then $y_1(t)$ is oscillatory by Lemma 6 and the preceding paragraph and is nonsingular by Lemma 2.

The last statement of the theorem is immediate from Lemma 3.

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