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A remark on a problem of Harary


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F. Harary [1] publicized the following question:

"For any graph G with p points, how can levels 1, 2, ..., p be assigned to the points in order to minimize the maximum of the absolute value of the differences between the levels of all pairs of adjacent points?"

We set

\[ \varphi(G) = \min \max |f(u) - f(v)| \]

where the maximum is taken over all edges (lines) uv and the minimum over all 1 - 1 mappings (valuations) \( f : V(G) \to \{1, 2, \ldots, p\} \); by \( V(G) \) we mean a set of all vertices (points) of \( G \).

Otherwise, our notation follows Harary [2]. Particularly, we reserve: a letter \( p \) for number of points, \( q \) for number of lines, \( d \) for diameter, \( \kappa \) for connectivity, \( \beta_0 \) for point independence number, \( d_i \) for degrees of points (\( d_1 \leq d_2 \leq \ldots \leq d_p \)).

As in [3], we write \( P_p^k \) for the \( k^{th} \) power of the path \( P_p \), in which two points \( u, v \) of \( V(P_p) = \{1, 2, \ldots, p\} \) are adjacent if, and only if, \( 0 < |u - v| \leq k \).

**Theorem 1.** \( \varphi(G) \) is the smallest integer \( k \) such that \( G \subset P_p^k \).

**Proof.** Every inclusion \( f : G \to P_p^k \) induces a valuation \( f : V(G) \to \{1, 2, \ldots, p\} \) such that \( \max |f(u) - f(v)| \leq k \) and vice versa. Hence, \( \varphi(G) \leq k \) if and only if \( G \subset P_p^k \), q.e.d.

Theorem 1 reduces — from a theoretical viewpoint — the original question to an elementary problem of graph theory: Given a pair of graphs, is one of them a subgraph of the other? In practice, however, an answer to the last question becomes extremely difficult, even with aid of high-speed computers. Therefore, it does not seem likely that one could find an effective algorithm to determine a minimal valuation of \( G \). Nevertheless, Theorem 1 will enable us to determine some relations between \( \varphi \)
and the other fundamental invariants of graphs. For this purpose, we shall need the following lemma; its proof is straightforward and will be omitted.

**Lemma.** The invariants of the graph $P^k_p$ satisfy:

\[
q = (p - 1) + (p - 2) + \ldots + (p - k) = k \frac{2p - k - 1}{2},
\]

\[
d = \left\{ \frac{p - 1}{k} \right\}, \quad \varkappa = k, \quad \beta_0 = 1 + \left\lfloor \frac{p - 1}{k + 1} \right\rfloor = \left\lfloor \frac{p}{k + 1} \right\rfloor,
\]

\[
d_j = \min \left( p - 1, k + \left\lfloor \frac{j - 1}{2} \right\rfloor, 2k \right).
\]

**Theorem 2.** For any graph $G$, we have

\[
\varphi \geq \frac{p - 1}{2} + \sqrt{(2p - 1)^2 - 8q},
\]

\[
\varphi \geq \frac{p - 1}{d}, \quad \varphi \geq \varkappa, \quad \varphi \geq \frac{p}{\beta_0} - 1,
\]

\[
\varphi \geq \max_j \left( d_j - \left\lfloor \frac{j - 1}{2} \right\rfloor, \frac{d_j}{2} \right).
\]

**Proof.** By Theorem 1, $G \subseteq P^k_p$. Now, observe that $p(G_1) = p(G_2), G_1 \subset G_2$ implies $q(G_1) \leq q(G_2), d(G_1) \geq d(G_2), \varkappa(G_1) \leq \varkappa(G_2), \beta_0(G_1) \geq \beta_0(G_2)$ and $d_j(G_1) \leq \leq d_j(G_2)$ for each $j$. The rest follows by our Lemma.

**Theorem 3.** If $m \geq n > 0$ then $\varphi(K_{mn}) = \left\lfloor (m - 1)/2 \right\rfloor + n$ and a valuation $f : V(K_{mn}) \rightarrow \{1, 2, \ldots, m + n\}$ is minimal whenever

\[
f(M) = \left\{ 1, 2, \ldots, \left\lfloor \frac{m}{2} \right\rfloor \right\} \cup \left\{ \left\lfloor \frac{m}{2} \right\rfloor + n + 1, \ldots, m + n \right\}
\]

where $M$ is the independent set of $K_{mn}$ having $m$ points.

**Proof.** If $1 \in f(M), p \notin f(M)$ ($p = m + n$ here) then $\max |f(u) - f(v)| = p - 1 > \varphi$ and $f$ is not minimal (similarly for $1 \notin f(M), p \in f(N)$). Therefore a minimal valuation $f$ satisfies either

(i) $1 \in f(M), p \in f(M)$

or

(ii) $1 \notin f(M), p \notin f(M)$. 
In case (i), observe that
\[
\max |f(u) - f(v)| = \max (p - \min f(N), \max f(N) - 1).
\]
Now, obviously, \(f(N) = \{j + 1, j + 2, \ldots, j + n\}\) for a minimal \(f\). A simple computation supplies the best values
\[
j = \left\lfloor \frac{p - n}{2} \right\rfloor = \left\lceil \frac{m}{2} \right\rceil, \quad \max |f(u) - f(v)| = \left\lceil \frac{m + 1}{2} \right\rceil + n - 1.
\]
In case (ii), we get similarly
\[
\max |f(u) - f(v)| = \left\lceil \frac{n + 1}{2} \right\rceil + m - 1
\]
and the assumption \(m \geq n\) is in favour of (i), q.e.d.

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References


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