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A REMARK ON A PROBLEM OF HARARY

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F. HARARY [1] publicized the following question:

"*For any graph G with p points, how can levels* 1, 2, …, *p be assigned to the points in order to minimize the maximum of the absolute value of the differences between the levels of all pairs of adjacent points?*"

We set

$$\varphi(G) = \min \max |f(u) - f(v)|$$

where the maximum is taken over all edges (lines) uv and the minimum over all $1-1$ mappings (valuations) $f : V(G) \to \{1, 2, \ldots, p\}$; by $V(G)$ we mean a set of all vertices (points) of G.

Otherwise, our notation follows Harary [2]. Particularly, we reserve: a letter p for number of points, q for number of lines, d for diameter, $\varkappa$ for connectivity, β_0 for point independence number, d_i for degrees of points $(d_1 \leqq d_2 \leqq \ldots \leqq d_p)$.

As in [3], we write P_p^k for the k^{th} power of the path P_p, in which two points u, v of $V(P_p) = \{1, 2, \ldots, p\}$ are adjacent if, and only if, $0 < |u - v| \leqq k$.

Theorem 1. *$\varphi(G)$ is the smallest integer k such that $G \subset P_p^k$.*

Proof. Every inclusion $f : G \to P_p^k$ induces a valuation $f : V(G) \to \{1, 2, \ldots, p\}$ such that $\max |f(u) - f(v)| \leqq k$ and vice versa. Hence, $\varphi(G) \leqq k$ if and only if $G \subset P_p^k$, q.e.d.

Theorem 1 reduces — from a theoretical viewpoint — the original question to an elementary problem of graph theory: Given a pair of graphs, is one of them a subgraph of the other? In practice, however, an answer to the last question becomes extremely difficult, even with aid of high-speed computers. Therefore, it does not seem likely that one could find an effective algorithm to determine a minimal valuation of G. Nevertheless, Theorem 1 will enable us to determine some relations between φ

and the other fundamental invariants of graphs. For this purpose, we shall need the following lemma; its proof is straightforward and will be omitted.

Lemma. *The invariants of the graph* P_p^k *satisfy*:

$$q = (p-1) + (p-2) + \ldots + (p-k) = k\frac{2p-k-1}{2},$$

$$d = \left\{\frac{p-1}{k}\right\}, \quad \varkappa = k, \quad \beta_0 = 1 + \left[\frac{p-1}{k+1}\right] = \left\{\frac{p}{k+1}\right\},$$

$$d_j = \min\left(p-1,\ k + \left[\frac{j-1}{2}\right],\ 2k\right).$$

Theorem 2. *For any graph G, we have*

$$\varphi \geqq p - \frac{1 + \sqrt{((2p-1)^2 - 8q)}}{2},$$

$$\varphi \geqq \frac{p-1}{d}, \quad \varphi \geqq \varkappa, \quad \varphi \geqq \frac{p}{\beta_0} - 1,$$

$$\varphi \geqq \max_j \max\left(d_j - \left[\frac{j-1}{2}\right], \frac{d_j}{2}\right).$$

Proof. By Theorem 1, $G \subset P_k^\varphi$. Now, observe that $p(G_1) = p(G_2)$, $G_1 \subset G_2$ implies $q(G_1) \leqq q(G_2)$, $d(G_1) \geqq d(G_2)$, $\varkappa(G_1) \leqq \varkappa(G_2)$, $\beta_0(G_1) \geqq \beta_0(G_2)$ and $d_j(G_1) \leqq d_j(G_2)$ for each j. The rest follows by our Lemma.

Theorem 3. *If* $m \geqq n > 0$ *then* $\varphi(K_{mn}) = [(m-1)/2] + n$ *and a valuation* $f : V(K_{mn}) \to \{1, 2, \ldots, m+n\}$ *is minimal whenever*

$$f(M) = \left\{1, 2, \ldots, \left[\frac{m}{2}\right]\right\} \cup \left\{\left[\frac{m}{2}\right] + n + 1, \ldots, m + n\right\}$$

where M is the independent set of K_{mn} *having m points.*

Proof. If $1 \in f(M)$, $p \notin f(M)$ ($p = m + n$ here) then $\max |f(u) - f(v)| = p - 1 > \varphi$ and f is not minimal (similarly for $1 \notin f(M)$, $p \in f(N)$). Therefore a minimal valuation f satisfies either

(i) $1 \in f(M)$, $p \in f(M)$

or

(ii) $1 \notin f(M)$, $p \notin f(M)$.

In case (i), observe that

$$\max |f(u) - f(v)| = \max (p - \min f(N), \max f(N) - 1) .$$

Now, obviously, $f(N) = \{j + 1, j + 2, \dots, j + n\}$ for a minimal f. A simple computation supplies the best values

$$j = \left[\frac{p - n}{2}\right] = \left[\frac{m}{2}\right], \quad \max |f(u) - f(v)| = \left[\frac{m + 1}{2}\right] + n - 1 .$$

In case (ii), we get similarly

$$\max |f(u) - f(v)| = \left[\frac{n + 1}{2}\right] + m - 1$$

and the assumption $m \geqq n$ is in favour of (i), q.e.d.

I thank Professor HARARY for his very valuable comments.

References

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[3] *F. Harary, R. M. Karp* and *W. T. Tutte:* A Criterion for Planarity of the Square of a Graph, J. Combinatorial Theory *2* (1967), 395—405.

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