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ON THE ACCESSIBILITY OF CONTROL SYSTEM  $\dot{x} \in Q(x)$ 

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In this paper we present an algebraic condition under which the set of all points which are reachable from a fixed point  $\omega$  at a constant time along solutions of a system (1) is a closed manifold whose dimension depends only on algebraic properties of  $\omega$ . At the same time we present an explicit formula for this manifold.

**Notations.**  $E_n$  denotes a Euclidean  $n$ -dimensional space with a norm  $\|\cdot\|$ . The dimension of a (finite dimensional) vector space  $V$  is written  $\dim V$ .  $\{p \in P; P(p)\}$  is the set of all points  $p \in P$  with property  $P(p)$ . We use only Lebesgue measures and integrals.

In the space  $\mathfrak{C}_n$  of all  $n$ -by- $n$  matrices we define a "bracket" operation  $[A, B] = BA - AB$ ,  $A, B \in \mathfrak{C}_n$ , which makes  $\mathfrak{C}_n$  a Lie algebra. Remind,  $L$  is a Lie algebra if it is a linear space with a bilinear anticommutative operation  $[\cdot, \cdot] : L \times L \rightarrow L$  such that

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0, \quad A, B, C \in L.$$

For  $A_1, A_2, \dots, A_r \in \mathfrak{C}_n$  we write  $[A_1, A_2, \dots, A_r] = [A_1, [A_2, \dots, [A_{r-1}, A_r] \dots]]$ . We often meet a matrix  $[A_1, A_2, \dots, A_r]$ , where  $A_1 = A_2 = \dots = A_{r-1}$ . Then in the case that there is no danger of misunderstanding we write it simply  $[A_1^{r-1} A_r]$ . Zero matrix is denoted by  $O$ , unit matrix by  $I$ , and the inverse of a nonsingular matrix  $A$  by  $A^{-1}$ .

A connected set  $S \subset E_n$  is called an  $r$ -dimensional manifold if for each  $x \in S$  there is an open nonempty set  $G \subset E_r$  and an injection  $\varphi : G \rightarrow S$  such that:

1.  $x \in \varphi(G)$ ,
2.  $\varphi(G)$  is open in  $S$ ,
3. Jacobian  $\partial\varphi/\partial t$  is continuous and has rank  $r$  on  $G$ .

Given an  $r$ -dimensional manifold  $S \subset E_n$  then the closure of  $S$  is called an  $r$ -dimensional closed manifold. A set  $S \subset E_n$  which contains only one element is said to be a 0-dimensional manifold.

**Definition 1.** Let  $\mathfrak{B} \subset \mathfrak{E}_n$  be a linear space. Then a mapping  $V$  defined on  $E_n$  by  $V(x) = \{Ax; A \in \mathfrak{B}\}$  is called a linear distribution created by  $\mathfrak{B}$ . If among all linear subspaces of  $\mathfrak{E}_n$  which create the same linear distribution  $V$  at least one is closed with respect to the bracket operation then we call  $V$  involutive.

**Definition 2.** Let  $V$  be a linear distribution and  $S \subset E_n$  a manifold. If for each  $x \in S$  the tangent space  $T(x)$  to  $S$  at  $x$  equals to  $V(x)$  then we call  $S$  an integral manifold of  $V$ .

Let  $V$  be a linear distribution. It was shown in [3] that then each  $x \in E_n$  is contained in an integral manifold of  $V$  if and only if  $V$  is involutive. Moreover, if  $V$  is involutive then each  $x \in E_n$  is contained in a unique integral manifold  $M_x$  of  $V$  which is maximal in the sense that any integral manifold  $M$  of  $V$  containing  $x$  is contained in  $M_x$ . Furthermore, let  $V$  be created by a Lie algebra  $\mathfrak{B}$  and let  $P_i \in \mathfrak{B}$ ,  $i = 1, 2, \dots, k$ , be chosen so that  $P_1x, P_2x, \dots, P_kx$  form a base of  $V(x)$ . Then there exists an open set  $G \subset E_k$  such that the mapping  $\varphi(t) = e^{P_1t_1} e^{P_2t_2} \dots e^{P_kt_k}x$ ,  $t \in G$ , describes an integral manifold  $\varphi(G)$  of  $V$ .

**Formulation of the problem.** Let us have a compact, convex set  $\mathfrak{U} \subset \mathfrak{E}_n$ . Denote  $Q(x) = \{Ax; A \in \mathfrak{U}\}$ ,  $x \in E_n$ . Then for each  $x \in E_n$   $Q(x)$  is compact and convex and the mapping  $Q(\cdot)$  is continuous on  $E_n$  if we equip the image of  $Q$  with Hausdorff topology. Hence existence of solutions of an equation

$$(1) \quad \dot{x} \in Q(x), \quad x(0) = \omega,$$

makes no problem. By a solution of (1) we mean any vector function  $x(\cdot)$ , absolutely continuous on an interval  $J \subset E_n$ , which fulfils  $\dot{x}(t) \in Q(x(t))$  for almost all  $t \in J$ .

Denote  $\mathcal{U} = \{u : [0, \infty) \rightarrow \mathfrak{U}; u \text{ measurable}\}$ . Then to each  $u \in \mathcal{U}$  and  $\omega \in E_n$  it corresponds a unique solution of an equation

$$(2) \quad \dot{x} = ux, \quad x(0) = \omega.$$

Without any ambiguity we denote this solution by  $x(\cdot, u, \omega)$ . According to implicit function theorem [4] for any solution  $x(\cdot)$  of (1) there exists  $u \in \mathcal{U}$  such that  $x(t) = x(t, u, \omega)$ ,  $t \in J$ .

Let  $\mathfrak{B}$  be the smallest Lie algebra which contains  $\mathfrak{U}$  and  $\mathfrak{W}$  the smallest linear space which contains the set  $\mathfrak{B} = \{A - B; A, B \in \mathfrak{U}\}$  and is closed with respect to bracket multiplication by elements from  $\mathfrak{B}$ . Then evidently  $\mathfrak{W} \subset \mathfrak{B}$  and  $\mathfrak{W}$  is a Lie algebra. Hence, mappings  $V$  and  $\mathcal{V}$ , defined by  $V(x) = \{Ax; A \in \mathfrak{B}\}$ ,  $\mathcal{V}(x) = \{Bx; B \in \mathfrak{W}\}$ ,  $x \in E_n$ , are involutive linear distributions.

For a given  $T \geq 0$  write  $\mathcal{S}_\omega(T) = \{x(T, u, \omega); u \in \mathcal{U}\}$  and  $S_\omega(T) = \bigcup_{t \in [0, T]} \mathcal{S}_\omega(t)$ . According to [3] the reachable cone  $\bigcup_{T \geq 0} S_\omega(T)$  of (1) is contained in the maximal integral manifold of  $V$  which passes through  $\omega$ . We are now looking for a condition

which guarantees that for any  $T > 0$  the set  $S_\omega(T)$ , resp.  $\mathcal{S}_\omega(T)$ , is a closure of an integral manifold of  $V$ , resp.  $\mathcal{V}$ .

**Auxiliaries. Lemma 1.** Denote by  $Z_u$  the fundamental matrix solution of (2), corresponding to  $u \in \mathcal{U}$ , for which  $Z_u(0) = I$ . Then for any  $A \in \mathfrak{B}$  and any  $t \geq 0$  we have  $Z_u(t) A Z_u^{-1}(t) - A \in \mathfrak{B}$  and  $Z_u^{-1}(t) A Z_u(t) - A \in \mathfrak{B}$ .

*Proof.* If  $u(\cdot)$  is piecewise constant then the assertion of Lemma 1 follows immediately from an identity  $e^{-C} B e^C = \sum_{k=0}^{\infty} (1/k!) \underbrace{[C, C, \dots, C, B]}_{k\text{-times}}$  which holds for any  $B, C \in \mathfrak{E}_n$ .

If  $u(\cdot)$  is not piecewise constant then we take a sequence  $u_k \in \mathcal{U}$ ,  $k = 1, 2, \dots$ , of piecewise constant functions which converges to  $u$  locally asymptotically on  $[0, \infty)$ . For any  $k$  and any  $t \geq 0$  we have  $Z_{u_k}(t) A Z_{u_k}^{-1}(t) - A \in \mathfrak{B}$ ,  $Z_{u_k}^{-1}(t) A Z_{u_k}(t) - A \in \mathfrak{B}$ . The sequence  $Z_{u_k}$ , resp.  $Z_{u_k}^{-1}$ ,  $k = 1, 2, \dots$ , converges to  $Z_u$ , resp.  $Z_u^{-1}$ , locally uniformly on  $[0, \infty)$ . The space  $\mathfrak{B}$  is finite dimensional, hence it is closed and the proof is complete.

*Remark.* We can similarly show that for  $B \in \mathfrak{B}$ ,  $u \in \mathcal{U}$  and  $t \geq 0$  it holds  $Z_u(t) \cdot B Z_u^{-1}(t) \in \mathfrak{B}$  and  $Z_u^{-1}(t) B Z_u(t) \in \mathfrak{B}$ .

**Lemma 2.**  $\dim \mathcal{V}(x(t, u, \omega)) = \dim \mathcal{V}(\omega)$  for any  $t \geq 0$  and any  $u \in \mathcal{U}$ .

*Proof.* According to [3] all points  $x(t, u, \omega)$  are contained in the maximal integral manifold of  $V$  which passes through  $\omega$ . Therefore it suffices to prove an equivalence  $A\omega \in \mathcal{V}(\omega)$  iff  $A x(t, u, \omega) \in \mathcal{V}(x(t, u, \omega))$ , where  $A \in \mathfrak{B}$ ,  $u \in \mathcal{U}$  and  $t \geq 0$ .

Fix  $A \in \mathfrak{B}$ ,  $u \in \mathcal{U}$  and  $t \geq 0$ . Put, for brevity,  $B = Z_u(t) A Z_u^{-1}(t) - A$ . Then we can write  $A\omega = Z_u^{-1}(t) (Z_u(t) A Z_u^{-1}(t) - A + A) Z_u(t) \omega = Z_u^{-1}(t) (B + A) Z_u(t) \omega$ .

Assume  $A\omega \in \mathcal{V}(\omega)$ . As  $B \in \mathfrak{B}$  we have  $Z_u^{-1}(t) B Z_u(t) \omega \in \mathfrak{B}$  and  $Z_u^{-1}(t) A Z_u(t) \omega = A\omega - Z_u^{-1}(t) B Z_u(t) \omega \in \mathcal{V}(\omega)$ . This implies existence of such  $B_i \in \mathfrak{B}$  and real numbers  $b_i$ ,  $i = 1, 2, \dots, p$ , that  $Z_u^{-1}(t) A Z_u(t) \omega = \sum_{i=1}^p b_i B_i \omega$ . Hence  $Ax(t, u, \omega) = Z_u(t) Z_u^{-1}(t) A Z_u(t) \omega = Z_u(t) \sum_{i=1}^p b_i B_i \omega = \sum_{i=1}^p b_i (Z_u(t) B_i Z_u^{-1}(t)) x(t, u, \omega) \in \mathcal{V}(x(t, u, \omega))$ , due to the remark to Lemma 1.

The inverse implication can be obtained by the same way.

Denote  $\mathcal{U}_0$  the set of all  $u \in \mathcal{U}$  which are piecewise continuous on  $[0, \infty)$  and moreover at each point of discontinuity continuous from the right. Till the end of this paragraph fix  $T > 0$  and  $u_0 \in \mathcal{U}_0$ . For any  $v \in \Delta = \{u - u_0; u \in \mathcal{U}\}$  and any  $\varepsilon \in [0, 1]$  we have  $u_\varepsilon = u_0 + \varepsilon v \in \mathcal{U}$ . The solution  $(x_\cdot, u_\varepsilon, \omega)$  of (2) is analytically dependent on  $\varepsilon$  and can be expanded into a power series

$$(3) \quad x(\cdot, u_\varepsilon, \omega) = \sum_{k=0}^{\infty} \varepsilon^k x_k(\cdot, v),$$

where the coefficients  $x_k(\cdot, v)$  solve an equation

$$(4) \quad \dot{x}_k = u_0 x_k + v x_{k-1}, \quad x_k(0, v) = 0, \quad k = 1, 2, \dots$$

Here we write for brevity  $x_0(t, v) = x(t, u_0, \omega)$ .

If we put  $a = \max \{ \|A\|; A \in \mathfrak{A} \}$  then  $\|x_0(t, v)\| \leq \|\omega\| e^{at}$  and for  $k = 1, 2, \dots$ , we have  $(d/dt) \|x_k\| \leq a \|x_k\| + 2a \|x_{k-1}\|$ , which implies  $\|x_k(t, v)\| \leq 2a \int_0^t e^{a(t-\tau)} \cdot \|x_{k-1}\| d\tau$ . Finally  $\|x_k(t, v)\| \leq \|\omega\| (2a)^k (1/k!) t^k e^{at}$ ,  $t \geq 0$ . Hence  $\|x(t, u_0, \omega)\| \leq \|\omega\| \sum_{k \geq 0} \varepsilon^k \|x_k(t, v)\| \leq \|\omega\| e^{3at}$  and the series (3) is locally uniformly absolutely convergent on  $[0, \infty)$ .

Using the variation of constants formula we get  $x_1(T, v) = Z(T) \int_0^T Z^{-1}(t) \cdot v(t) x(t, u_0, \omega) dt = Z(T) \int_0^T Z^{-1}(t) v(t) Z(t) dt \cdot Z^{-1}(T) x(T, u_0, \omega)$ , where we, for brevity, write  $Z$  instead of  $Z_{u_0}$ .

**Lemma 3.** *The linear hull  $\mathfrak{M}(T, u_0)$  of set  $\{ \int_0^T Z^{-1}(t) v(t) Z(t) dt; v \in \Delta \}$  equals to the linear hull of  $\{ Z^{-1}(t) B Z(t); t \in [0, T], B \in \mathfrak{B} \}$ .*

*Proof.* Take  $B \in \mathfrak{B}$ . There are matrices  $A_{1,2} \in \mathfrak{A}$  such that  $B = A_1 - A_2$ . As  $\mathfrak{A}$  is convex we have  $u_i(t) = \frac{1}{2}(A_i + u_0(t)) \in \mathfrak{U}$ ,  $i = 1, 2$ , and  $v_i = u_i - u_0 \in \Delta$ ,  $i = 1, 2$ . The function  $u_0 \in \mathfrak{U}_0$  is everywhere continuous from the right therefore  $v_{1,2}$  are continuous from the right too.

Take  $t_0 \in [0, T)$  and for any  $\alpha \in (0, T - t_0)$  denote

$$v_{i,\alpha}(t) = \begin{cases} v_i(t) & \text{for } t \in [t_0, t_0 + \alpha) \\ 0 & \text{for } t \notin [t_0, t_0 + \alpha) \end{cases}, \quad i = 1, 2.$$

Then  $v_{i,\alpha} \in \Delta$ . Due to continuity from the right of functions  $v_i$  we get

$$\lim_{\alpha \rightarrow 0+} \frac{1}{\alpha} \int_0^T Z^{-1}(t) v_{i,\alpha}(t) Z(t) dt = Z^{-1}(t_0) v_i(t_0) Z(t_0) \in \mathfrak{M}(T, u_0)$$

and

$$Z^{-1}(t_0) B Z(t_0) = 2 Z^{-1}(t_0) (v_1(t_0) - v_2(t_0)) Z(t_0) \in \mathfrak{M}(T, u_0).$$

Finally

$$\lim_{t_0 \rightarrow T-} Z^{-1}(t_0) B Z(t_0) = Z^{-1}(T) B Z(T) \in \mathfrak{M}(T, u_0).$$

The inverse inclusion follows immediately from the fact that the values of any  $v \in \Delta$  lie in  $\mathfrak{B}$ .

**Definition.** We say that a compact convex set  $\mathfrak{A} \subset \mathfrak{C}_n$  has a property (A) if such matrices  $A_i \in \mathfrak{A}$ ,  $i = 1, 2, \dots, p$ , and  $B_j \in \mathfrak{B}$ ,  $j = 1, 2, \dots, q$ , exists that the linear space generated by matrices  $[A_i^r B_j]$ ,  $i = 1, 2, \dots, p$ ,  $j = 1, 2, \dots, q$ ,  $r = 0, 1, 2, \dots$ , equals to the Lie algebra  $\mathfrak{B}$ .

**Lemma 4.** *If  $\mathfrak{A} \subset \mathfrak{C}_n$  has property (A) then for any  $T > 0$  there exists such  $u_T \in \mathcal{U}_0$  that  $\mathfrak{M}(T, u_T) = \mathfrak{B}$ .*

*Proof.* Take  $T > 0$  and a partition  $0 = t_0 < t_1 < \dots < t_{p+1} = T$  of interval  $[0, T]$ . Choose an arbitrary matrix from  $\mathfrak{A}$  and denote it by  $A_{p+1}$ . Define  $u(t) = A_i, t \in [t_{i-1}, t_i], i = 1, 2, \dots, p+1, u(T) = A_{p+1}$ . Evidently  $u \in \mathcal{U}_0$  and  $Z_u(t) = e^{A_i(t-t_{i-1})} e^{A_{i-1}(t_{i-1}-t_{i-2})} \dots e^{A_1(t_1-t_0)}$ , for  $t \in [t_{i-1}, t_i]$ .

For  $t \in [t_{i-1}, t_i]$  and  $B \in \mathfrak{B}$  we have

$$\begin{aligned} Z_u^{-1}(t) B Z_u(t) &= Z_u^{-1}(t_{i-1}) e^{-A_i(t-t_{i-1})} B e^{A_i(t-t_{i-1})} Z_u(t_{i-1}) = \\ &= Z_u^{-1}(t_{i-1}) \sum_{r=0}^{\infty} \frac{(t-t_{i-1})^r}{r!} [A_i^r B] Z_u(t_{i-1}). \end{aligned}$$

By differentiation with respect to  $t$  we get  $Z_u^{-1}(t_{i-1}) [A_i^r B] Z_u(t_{i-1}) \in \mathfrak{M}(T, u), r = 0, 1, \dots$

If we take  $t_p$  sufficiently small then the dimension of a linear space generated by matrices  $Z_u^{-1}(t_{i-1}) [A_i^r B_j] Z_u(t_{i-1}), i = 1, 2, \dots, p, j = 1, 2, \dots, q, r = 0, 1, \dots$ , equals to the dimension of a linear space  $L$  generated by matrices  $[A_i^r B_j], i = 1, 2, \dots, j = 1, 2, \dots, q, r = 0, 1, \dots$ . It implies  $\dim \mathfrak{M}(T, u) \geq \dim L$ . Lut  $\mathfrak{M}(T, u) \subset \mathfrak{B}$  and we have assumed  $L = \mathfrak{B}$ . This gives us the desired equality  $\mathfrak{M}(T, u) = \mathfrak{B}$ .

**Lemma 5.** *Let  $A_0 \in \mathfrak{A}$  be an arbitrary matrix. Then  $\mathfrak{B}$  is a linear hull of  $A_0$  and  $\mathfrak{B}$ .*

*Proof.* One inclusion is trivial. As  $\mathfrak{B}$  is a linear hull of matrices of a type  $[A_1, A_2, \dots, A_k]$ , where  $A_i \in \mathfrak{A}, i = 1, 2, \dots, k$ , it suffices to show that each such matrix belongs to the linear hull of  $A_0$  and  $\mathfrak{B}$ . If  $k = 1$  then  $A_1 = A_0 + (A_1 - A_0)$ , where  $A_1 - A_0 \in \mathfrak{B}$ . If  $k > 1$  then  $[A_1, \dots, A_{k-1}, A_k] = [A_1, \dots, A_{k-1}, A_k - A_{k-1}] \in \mathfrak{B}$  which completes the proof.

**Main result.** *Let  $\mathfrak{A} \subset \mathfrak{C}_n$  have property (A). Construct distributions  $\mathcal{V}$  and  $V$ . For given  $T > 0$  and  $\omega \in E_n$  let  $\mathcal{S}_\omega(T)$ , resp.  $S_\omega(T)$ , be the set of all points which are reachable at the time  $T$ , resp. at any time  $t \in [0, T]$ , from  $\omega$  along solutions of (1). Denote  $\dim \mathcal{V}(\omega) = q$  and  $\dim V(\omega) = r$ .*

*Then  $\mathcal{S}_\omega(T)$ , resp.  $S_\omega(T)$ , is a closed  $q$ -, resp.  $r$ -, dimensional integral manifold of the distribution  $\mathcal{V}$ , resp.  $V$ .*

*Proof.* Let  $T > 0$ . Take  $x \in \mathcal{S}_\omega(T)$  and  $\varepsilon > 0$ . By implicit function theorem there exists such  $u \in \mathcal{U}$  that  $x = x(T, u, \omega)$ . According to Lemma 4 for any  $\lambda > 0$  there exists  $\varphi_\lambda \in \mathcal{U}_0$  such that  $\mathfrak{M}(\lambda, \varphi_\lambda) = \mathfrak{B}$ . Define  $u_\lambda(t) = \begin{cases} u(t), & t > \lambda \\ \varphi_\lambda(t), & t \in [0, \lambda] \end{cases}$ . As  $\mathfrak{A}$  is bounded the functions  $u_\lambda(\cdot)$  converge asymptotically to  $u(\cdot)$  on  $[0, T]$  with  $\lambda \rightarrow 0+$ .

Therefore  $\lim_{\lambda \rightarrow 0^+} x(T, u_\lambda, \omega) = x(T, u, \omega) = x$  and we can fix such  $\lambda_0$  that  $\|x(T, u_{\lambda_0}, \omega) - x\| < \varepsilon$ .

Denote, for brevity, by  $Z(\cdot)$  the fundamental matrix solution of (2), corresponding to  $u_{\lambda_0}$ . According to Lemma 3 there are functions  $v_i \in \{u - u_{\lambda_0}; u \in \mathcal{U}\}$ ,  $i = 1, 2, \dots, k$ , such that matrices  $\int_0^{\lambda_0} Z^{-1}(t) v_i(t) Z(t) dt$ ,  $i = 1, 2, \dots, k$ , form a base of  $\mathfrak{B}$ . Define  $w_i(t) = \begin{cases} v_i(t), & t \in [0, \lambda_0] \\ 0, & t > \lambda_0 \end{cases}$ ,  $i = 1, 2, \dots, k$ . Then evidently also  $w_i(t) \in \{u - u_{\lambda_0}; u \in \mathcal{U}\}$ .

The matrices  $B_i = Z(T) \int_0^T Z^{-1}(t) w_i(t) Z(t) dt Z^{-1}(T)$ ,  $i = 1, 2, \dots, k$ , are linearly independent and according to Lemma 1 they belong into  $\mathfrak{B}$ . Hence they form a base of  $\mathfrak{B}$  and vectors  $B_i x(T, u_{\lambda_0}, \omega)$ ,  $i = 1, 2, \dots, k$ , generate the linear space  $\mathcal{V}(x(T, u_{\lambda_0}, \omega))$ . According to Lemma 2 we have  $\dim \mathcal{V}(x(T, u_{\lambda_0}, \omega)) = \dim \mathcal{V}(\omega) = q$ . Assume that vectors  $B_i x(T, u_{\lambda_0}, \omega)$ ,  $i = 1, 2, \dots, q$ , form a base of  $\mathcal{V}(x(T, u_{\lambda_0}, \omega))$ .

Now

$$(5) \quad x(T, u_{\lambda_0} + \sum_{i=1}^q \tau_i w_i, \omega), \quad \text{where } \tau \in G = \left\{ \tau \in E_q; \sum_{i=1}^q |\tau_i| < 1 \right\},$$

represents a mapping of an open set  $G \subset E_q$  into  $\mathcal{S}_\omega(T)$ . It has continuous first partial derivatives with respect to  $\tau$ , which are for  $\tau = 0$  solutions of a corresponding equation

$$(6) \quad \frac{d}{dt} \frac{\partial x}{\partial \tau_i} = u_{\lambda_0} \frac{\partial x}{\partial \tau_i} + w_i x(t, u_{\lambda_0}, \omega), \quad \left. \frac{\partial x}{\partial \tau_i} \right|_{\tau=0} = 0, \quad i = 1, 2, \dots, q.$$

If we compare (6) with (4) we see that the vectors  $B_i x(T, u_{\lambda_0}, \omega)$ ,  $i = 1, 2, \dots, q$ , are columns of Jacobian  $Dx/D\tau$  at  $\tau = 0$ . Therefore there exists a neighborhood  $G_0 \subset G$  of origin in  $E_q$  such that  $Dx/D\tau$  has rank  $q$  at any point of  $G_0$ .

Denote  $X_\tau$  the fundamental matrix solution of (2), corresponding to  $u_{\lambda_0} + \sum_{i=1}^q \tau_i w_i$ , for which  $X_\tau(0) = I$ . Then we can write

$$\begin{aligned} \frac{\partial}{\partial \tau_i} x(T, u_{\lambda_0} + \sum_{i=1}^q \tau_i w_i, \omega) &= X_\tau(T) \int_0^T X_\tau^{-1}(t) \sum_{i=1}^q \tau_i w_i X_\tau(t) dt X_\tau^{-1}(T) \\ &\cdot x(T, u_{\lambda_0} + \sum_{i=1}^q \tau_i w_i, \omega) \in \mathcal{V}(x(T, u_{\lambda_0} + \sum_{i=1}^q \tau_i w_i, \omega)). \end{aligned}$$

Thus,  $\mathcal{S}_\omega(T)$  is a closure of a union of a family of  $q$ -dimensional integral manifolds of distribution  $\mathcal{V}$ .

Let  $\mathcal{S}_{0,1}$  be two manifolds of this family. Choose  $x(T, u_i, \omega) \in \mathcal{S}_i$ ,  $i = 0, 1$ , so that there exists  $t_0 \in (0, T]$  such that  $u_0(t) = u_1(t)$  for  $t \in [0, t_0]$  and  $\mathfrak{M}(t_0, u_0) = \mathfrak{B}$ . Denote  $u_\lambda = (1 - \lambda) u_0 + \lambda u_1$ ,  $\lambda \in [0, 1]$ . Then the curve  $\Gamma(\lambda) = x(T, u_\lambda, \omega)$ ,  $\lambda \in [0, 1]$  links points  $x(T, u_i, \omega)$ ,  $i = 0, 1$ , and each its point is contained in  $\mathcal{S}_\omega(T)$ .

Let us prove that  $d\Gamma(\lambda)/d\lambda \in \mathcal{V}(\Gamma(\lambda))$  for  $\lambda \in [0, 1]$ . It holds

$$\frac{d}{dt} \frac{dx(t, u_\lambda, \omega)}{d\lambda} = u_\lambda(t) \frac{dx(t, u_\lambda, \omega)}{d\lambda} + (u(t) - u_0(t)) x(t, u_\lambda, \omega), \quad \left. \frac{dx(t, u_\lambda, \omega)}{d\lambda} \right|_{t=0} = 0.$$

Using variation of constants formula we get

$$\begin{aligned} \frac{d\Gamma(\lambda)}{d\lambda} &= \frac{dx(T, u_\lambda, \omega)}{d\lambda} = \\ &= Z_{u_\lambda}(T) \int_0^T Z_{u_\lambda}^{-1}(t) (u(t) - u_0(t)) Z_{u_\lambda}(t) dt Z_{u_\lambda}^{-1}(T) x(T, u_\lambda, \omega) \in \mathcal{V}(\Gamma(\lambda)), \quad \lambda \in [0, 1]. \end{aligned}$$

We have proved that  $\Gamma(\lambda)$ ,  $\lambda \in [0, 1]$ , is contained in the maximal integral manifold  $\mathcal{M}$  of  $\mathcal{V}$  which passes through  $x(T, u_0, \omega)$ . As  $\mathcal{S}_\lambda \cap \mathcal{M} \neq \emptyset$  for any  $\lambda \in [0, 1]$  it follows from [3] that  $\mathcal{S}_\lambda \subset \mathcal{M}$ ,  $\lambda \in [0, 1]$ . Hence  $\mathcal{S} = \bigcup_{\lambda \in [0, 1]} \mathcal{S}_\lambda \subset \mathcal{S}_\omega(T)$  is an integral manifold of  $\mathcal{V}$  which contains both points  $x(T, u_i, \omega)$ ,  $i = 0, 1$ . First part of the theorem is proved.

Take  $x \in S_\omega(T)$  and  $\varepsilon > 0$ . Again there exists  $u \in \mathcal{U}$  and  $t_0 \in [0, T]$  such that  $x = x(t_0, u, \omega)$ . By the same way we find functions  $u_0 \in \mathcal{U}$ ,  $w_i \in \{u - u_0; u \in \mathcal{U}\}$ ,  $i = 1, 2, \dots, q$ , a number  $t_1 > 0$  and a neighborhood  $G \subset E_q$  of origin such that Jacobian of a mapping  $x(t_1, u_0 + \sum_{i=1}^q \tau_i w_i, \omega) : G \rightarrow E_n$  has rank  $q$  at any point of  $G$ , and moreover  $u_0(t_1) \neq 0$  and  $\|x(t_1, u_0, \omega) - x\| < \varepsilon$ .

If  $\dim V(\omega) = q$  there is nothing to be proved. Assume  $\dim V(\omega) > q$ . Investigate a mapping

$$(7) \quad x(t, u_0 + \sum_{i=1}^q \tau_i w_i, \omega), \quad |t - t_1| < \delta, \quad \tau \in G.$$

We know that for any  $i = 1, 2, \dots, q$ ,

$$\frac{\partial}{\partial \tau_i} x(t, u_0 + \sum_{i=1}^q \tau_i w_i, \omega) \in \mathcal{V}(x(t, u_0 + \sum_{i=1}^q \tau_i w_i, \omega)).$$

But

$$\begin{aligned} &\frac{\partial}{\partial t} x(t, u_0 + \sum_{i=1}^q \tau_i w_i, \omega) = \\ &= (u_0 + \sum_{i=1}^q \tau_i w_i) x(t, u_0 + \sum_{i=1}^q \tau_i w_i, \omega) \in V(x(t, u_0 + \sum_{i=1}^q \tau_i w_i, \omega)). \end{aligned}$$

If we take  $\delta$  and an open set  $G_0 \subset G$ ,  $0 \in G_0$ , so small that  $A(t, \tau) = u_0(t) + \sum_{i=1}^q \tau_i w_i(t) \neq 0$  for  $t \in (t_1 - \delta, t_1 + \delta)$ ,  $\tau \in G_0$ , then according to Lemma 5 the space  $\mathfrak{B}$  is equal to the linear hull of  $\mathfrak{B}$  and  $A(t, \tau)$  for any  $t \in (t_1 - \delta, t_1 + \delta)$ ,  $\tau \in G_0$ .



Hence Jacobian of the mapping (7) has at any point from  $(t_1 - \delta, t_1 + \delta) \times G_0$  rank  $r$ .

Similarly as in the first part we find out that  $S_\omega(T)$  is a closure of a union of a family of  $r$ -dimensional integral manifolds of the distribution  $V$ . Let again we have two manifolds  $S_0, S_1$ , of this family. Then we can choose points  $x(t_i, u_i, \omega) \in S_i, i = 0, 1$ , so that  $t_2 = \min(t_0, t_1) > 0$  and there exists  $t_3 \in (0, t_2]$  such that  $u_0(t) = u_1(t)$  for  $t \in [0, t_3]$  and  $\mathfrak{M}(t_3, u_0) = \mathfrak{B}$ . The case  $t_0 = t_1$  has been already treated, therefore assume  $t_0 < t_1$ . Then a curve  $\Gamma$  consisting of arches  $\Gamma_0(t) = x(t, u_0, \omega), t \in [t_0, t_1]$ , and  $\Gamma_1(\lambda) = x(t_1, (1 - \lambda)u_0 + \lambda u_1, \omega), \lambda \in [0, 1]$ , again links points  $x(t_i, u_i, \omega), i = 0, 1$ , is contained in  $S_\omega(T)$ , and through each of its points there passes an integral manifold of  $V$ . Hence the union of these manifolds is again an integral manifold of  $V$ , contains  $S_0$  and  $S_1$ , and is contained in  $S_\omega(T)$ . Q.E.D.

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