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ON THE COMPARATIVE GROWTH OF CONVEX FUNCTIONS

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1. Introduction. All functions that we consider below are real valued functions defined on $(0, \infty)$, and are non-negative. Suppose $\varphi(u)$ is an absolutely continuous and increasing function for $0 < u < \infty$. Observe $\varphi'(u)$ exists almost everywhere in $(0, \infty)$ and is > 0. We use the notation $M(u) \# \varphi(u)$ to mean that "*$M(u)$ is a convex function with respect to $\varphi(u)$*" in $(0, \infty)$. Using the classical definition of convexity, it can be proved that (see p. 73, (4), [1])

$$\text{(A)} \qquad M(u) \# \varphi(u) \quad \text{in} \quad (0, \infty) \Leftrightarrow M(u_2) = M(u_1) + \int_{u_1}^{u_2} n(x) \, \mathrm{d}\varphi(x) \,,$$

where $u_1, u_2 \in (0, \infty)$, $u_2 > u_1$ monotonically increasing and the integral is taken in the Lebesgue sense. An important consequence of (A) is that $\lim (M(u)/\varphi(u)) \to \infty$ as $u \to \infty$. But $\log M(u)/\varphi(u)$ may or may not tend to ∞ (take $f_1(z) = \exp(\exp(z))$, $f_2(z) = \exp(z)$, $z = re^{i\theta}$; $M(r) = \log M_i(r)$, $\varphi(r) = \log r$ where $M_i(r) = \sup_{|z|=r} |f_i(z)|$) and even more, the limit of this function may not even exist (see for instance Example 1, [6] and take $\varphi(\sigma) = \sigma$, $M(\sigma) = \log \mu(\sigma)$). We therefore naturally think of defining

$$\text{(B)} \qquad \overline{\underline{\lim}}_{u \to \infty} \frac{\log M(u)}{\varphi(u)} = \frac{A}{B}\,; \quad 0 \leqq B \leqq A \leqq \infty \,.$$

In this note we introduce another function in terms of $M(u)$ and $\varphi(u)$ and characterizes its all important properties similar to (A), (B) and its asympotic relation with $M(u)$. These results are collected in the form of lemmas which together yield a number of interesting results as particular cases in the theory of entire functions (§ 3 where we merely sketch the applications and details are left to the reader).

2. A new dependent convex function. To get our results precisely, let us introduce a function $N : R \to R$, $R = \{u : 0 < u < \infty\}$, by the following equality

$$\text{(C)} \qquad \exp(N(u) + k\,\varphi(u)) = \int_0^u \exp(M(x) + k\,\varphi(x))\,\varphi'(x)\,\mathrm{d}x\,, \quad 0 < k < \infty\,,$$

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where k is a constant. Then the following results hold between $M(u)$ and $N(u)$, all being collected in the form of six lemmas.

Lemma 1. $\exp(M(u) + k\,\varphi(u)) \# \exp(N(u) + k\,\varphi(u))$ *in* $(0, \infty)$.

Proof. We have

$$\frac{\mathrm{d}\exp(M(u) + k\,\varphi(u))}{\mathrm{d}\exp(N(u) + k\,\varphi(u))} = \frac{\mathrm{d}\exp(M(u) + k\,\varphi(u))}{\exp(M(u) + k\,\varphi(u))\,\mathrm{d}\varphi(u)} =$$
$$= \frac{\mathrm{d}(M(u) + k\,\varphi(u))}{\mathrm{d}\varphi(u)} = k + n(u) = p(u)\,,$$

say. Hence

$$\exp(M(u) + k\,\varphi(u)) = \exp(M(\alpha) + k\,\varphi(\alpha)) + \int_\alpha^u p(u)\,\mathrm{d}\exp(N(u) + k\,\varphi(u)),\ \alpha > 0$$
$$\Rightarrow \text{by (A)},\ \exp(M(u) + k\,\varphi(u)) \# \exp(N(u) + k\,\varphi(u))\,.$$

Lemma 2. *This is,* $N(u) \# \varphi(u)$ *in* (u_0, ∞), *for some large* u_0.

Proof. We have

$$\exp(N(u) + k\,\varphi(u))(\mathrm{d}N(u) + k\,\mathrm{d}\varphi(u)) = \exp(M(u) + k\,\varphi(u))\,\mathrm{d}\varphi(u)\,.$$

Hence

$$\mathrm{d}N(u) = \exp(M(u) - N(u))\,\mathrm{d}\varphi(u) - k\,\mathrm{d}\varphi(u)\,.$$

But by Lemma 1, $\exp(M(u) - N(u))$ is monotonically increasing and as k is already a fixed number, we can choose an u_0 in $(0, \infty)$, such that $\exp(M(u) - N(u)) - k$ is monotonically increasing in (u_0, ∞). Consequently, if we write $\exp(M(u) - N(u)) - k$ as $r(u)$, then $r(u)$ is monotonically increasing in (u_0, ∞), and the result follows from (A).

Lemma 3. *Let*

$$\varlimsup_{u\to\infty} \frac{\log M(u)}{\varphi(u)} = \frac{A}{B}\,;$$

Then

$$\varlimsup_{u\to\infty} \frac{\log N(u)}{\varphi(u)} = \frac{A}{B}\,.$$

Proof. From that the fact that $M(u) \# \varphi(u)$, it follows that $M(u_2) \geqq M(u_1)$ for $u_2 > u_1$, and so

$$\exp(N(u) + k\,\varphi(u)) \leqq \exp(M(u)) \int_0^u \exp(k\,\varphi(x))\,\mathrm{d}\varphi(x) \leqq \frac{1}{k}\exp(M(u) + k\,\varphi(u))$$

or

(D) $$N(u) \leqq (1 + o(1))\, M(u)\,, \quad \text{as} \quad M(u) \to \infty \quad \text{with } u\,.$$

Further, if $\vartheta > u > 0$, then

$$\exp\left(N(\vartheta) + k\,\varphi(\vartheta)\right) \geqq \int_u^{\vartheta} \exp\left(M(x) + k\,\varphi(x)\right) \mathrm{d}\varphi(x) \geqq$$

$$\geqq \exp\left(M(u)\right) . \frac{1}{k} \left\{\exp\left(k\,\varphi(\vartheta)\right) - \exp\left(k\,\varphi(u)\right)\right\}$$

or,

$$N(\vartheta) \geqq M(u) + \log\left[1 - \exp\left\{k(\varphi(u) - \varphi(\vartheta))\right\}\right] - \log k\,.$$

Now choose $\vartheta > u$ such that $\varphi(\vartheta) - \varphi(u) \to L$ as $u \to \infty$, then for all $u \geqq u_0$

(E) $$\frac{\log N(\vartheta)}{\varphi(\vartheta)} \geqq (1 + O(1))\, \frac{\log M(u)}{\varphi(u)}\,.$$

Inequalities (D) and (E) furnish the proof.

Lemma 4. *If*

$$\overline{\lim_{u \to \infty}}\, \frac{\log N(u)}{\varphi(u)} = \frac{A}{B}\,;$$

then

$$\overline{\lim_{u \to \infty}}\, \frac{\log M(u)}{\varphi(u)} = \frac{A}{B}\,.$$

Proof. This is similar to the proof of lemma 3 and so omitted.

Lemmas 3 and 4 suggest that there is something common in $M(u)$ and $N(u)$ which give rise to the same limits A & B. One might expect a pair $(M(u), N(u))$ which will give the limits A and B. This expectation is accorded by

Lemma 5. *Let*

$$\overline{\lim_{u \to \infty}}\, \frac{\log M(u)}{\varphi(u)} = \frac{A}{B}\,;$$

then

(F) $$\overline{\lim_{u \to \infty}}\, \exp\left\{(M(u) - N(u))\, \frac{1}{\varphi(u)}\right\} = \frac{e^A}{e^B}\,.$$

Proof. It follows from the representation of $N(u)$ that

$$N(u) + k\,\varphi(u) = O(1) + \int_{u_0}^{u} \exp\left\{M(x) - N(x)\right\} \mathrm{d}\varphi(x) \leqq O(1) + \int_{u_0}^{u} (L + \varepsilon)^{\varphi(x)}\, \mathrm{d}\varphi(x)\,,$$

where L is the limit superior in (F). Now $N(u) \# \varphi(u)$ and so $\varphi(u) = o(N(u))$ and hence for all large u,

$$(1 + O(1))\, N(u) \leqq (1 + o(1)) \frac{\exp\{\log(L + \varepsilon)\, \varphi(u)\}}{\log(L + \varepsilon)}$$

and using Lemma 3, we obtain $L \geqq e^A$. Let now $\vartheta > u > 0$, such that $\varphi(\vartheta) - \varphi(u) \to \to C \geqq 0$ as $u \to \infty$. Now

$$N(\vartheta) + k\, \varphi(\vartheta) \geqq \int_u^\vartheta \exp\{M(x) - N(x)\}\, d\varphi(x) \geqq (L - \varepsilon)^u (\varphi(\vartheta) - \varphi(u)),$$

for a sequence of $u \to \infty$. Hence, as before $L \leqq e^A$ and this combined with a similar reverse inequality proves the first part of the lemma. The other part similarly follows.

From various results proved above for $M(u)$ and $N(u)$, it roughly appears that these two functions behave in a similar manner for large values of u, but what we can say about this question is contained in the following

Lemma 6. *Let A stand as in lemma 3 and suppose further that $A < \infty$, then $N(u) \sim M(u)$, $u \to \infty$.*

Proof. From (1) it follows that $\lim_{u \to \infty} N(u)/M(u) \leqq 1$ and we now therefore prove a reverse inequality of this limit. Let therefore $\vartheta > u \geqq u_0$. Then from the representation of $N(\vartheta)$ in terms of $N(u)$, we obtain

$$N(\vartheta) + k\, \varphi(\vartheta) \leqq N(u) + k\, \varphi(u) + \exp(M(\vartheta) - N(\vartheta))\, (\varphi(\vartheta) - \varphi(u)) \leqq$$
$$\leqq N(u) + k\, \varphi(u) + (\varphi(\vartheta) - \varphi(u))\, e^{(A+\varepsilon)\varphi(\vartheta)},$$

by Lemma 5. Choose ϑ, such that

$$\varphi(\vartheta) - \varphi(u) = e^{-(A+\varepsilon)\varphi(u)},$$

then

$$N(\vartheta) + k\, \varphi(\vartheta) \leqq N(u) + k\, \varphi(u) + \exp\{(A + \varepsilon)(\varphi(\vartheta) - \varphi(u))\} =$$
$$= N(u) + k\, \varphi(u) + \exp\{(A + \varepsilon) \exp(-(A + \varepsilon)\, \varphi(u))\}.$$

But from (C) it is easily seen that for any $\vartheta > u$

$$\exp(M(\vartheta) - k\, \varphi(v)) \geqq \frac{\exp(M(u))}{K} \{\exp(k\, \varphi(\vartheta)) - \exp(k\, \varphi(u))\},$$

and this inequality when combined with the preceding one and using the value of $\varphi(\vartheta) - \varphi(u)$ once again, gives $\lim_{u \to \infty} M(u)/N(u) \leqq 1$ and the result follows.

3. Applications. The above lemmas are extremely useful in proving certain results on various mean values in the theory of entire functions represented by Dirichlet and Taylor series. Consider first of all Dirichlet entire series, and suppose $\log A(\sigma) = M(\sigma)$, $\log M_K(\sigma) = \log N(\sigma)$ and $\varphi(\sigma) = \sigma$ where $A(\sigma)$ and $M_K(\sigma)$ are mean values as introduced in [4]. One then finds that many of the results of [4] and [5] are simple consequences of the lemmas of this note. Similarly, if $\log M_\delta(r) = M(r)$, $\log M_{\sigma o k}(r) = \log N(r)$ and $\varphi(r) = \log r$ where $M_\delta(r)$ and $M_{\delta,k}(r)$ are mean values of an entire Taylor series over $|Z| = r$, and $|Z| \leqq r$ (for these definitions and results depending on them, see [2], [3]), then a number of results, say for instance Th. 1 of [3] and, (ix) and (x) of [2] are easy consequences of lemmas 1 through 6. After going through these details cited above, one is inclined to think that these lemmas simultaneously give a number of results, as simple corollaries, in the theory of entire functions represented by Dirichlet and Taylor series and this is indeed an attractive feature of these results.

References

[1] *P. K. Kamthan:* Convex functions and their applications, Rev. Facul. Scs. Uni. d'Istan., Ser. A, *28* (1963), 71—78.

[2] *P. K. Kamthan:* On the mean values of an entire function, The Math. Student, Nos. 1 & 2, *32*, (1964), 101—109.

[3] *P. K. Kamthan:* Mean values of entire functions (II), Rev. Mate. His-Americana, Ser. 4[a], No. 6, *24*, (1964), 218—222.

[4] *P. K. Kamthan:* On the mean values of an entire function represented by Dirichlet series, Acta Math. Acad. Sci. Hung, *15*, (1964), 133—136.

[5] *P. K. Kamthan:* The above article (II), the above Journal, *16*, (1965), 89—92.

[6] *P. K. Kamthan:* On entire functions represented by Dirichlet series (IV), Ann. Inst. Fourier, Grenoble, *16*, (1966), 209—223.

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