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THE RICCATI DIFFERENTIAL EQUATION WITH COMPLEX-VALUED COEFFICIENTS

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In this paper we study the character of the solution of the system

$$\frac{\mathrm{d}u}{\mathrm{d}t} = a(t) - u^2 + v^2 \,, \qquad \frac{\mathrm{d}v}{\mathrm{d}t} = b(t) - 2uv \tag{1}$$

where $a(t)$, $b(t)$ are continuous functions on an interval $J = \langle t_0, \infty)$. The special case $a(t) \equiv 0$ of this system has appeared recently in a paper [1] by C. Kulig. There was generalized a theorem by Z. Butlewski [2], [3] about the trajectories of the system

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = r \left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2 ,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2 \frac{\mathrm{d}\varphi}{\mathrm{d}t}\right) = A(t)\, r^2$$

which can be transformed into (1) by $u = r'/r$, $v = \varphi'$.

Consider first the autonomous system

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \alpha - u^2 + v^2 \,, \qquad \frac{\mathrm{d}v}{\mathrm{d}t} = \beta - 2uv \,, \tag{2}$$

where α, β are real constants. The singular points (λ, μ) of this system satisfy the equations

$$\alpha - \lambda^2 + \mu^2 = 0 ,$$
$$\beta - 2\lambda\mu = 0 .$$

If $\alpha^2 + \beta^2 > 0$ we see that the system (2) has two singular points (λ, μ) and $(-\lambda, -\mu)$ where

$$(3) \qquad \lambda = \sqrt{[\tfrac{1}{2}(\sqrt{(\alpha^2 + \beta^2)} + \alpha)]} , \quad \mu = \pm\sqrt{[\tfrac{1}{2}(\sqrt{(\alpha^2 + \beta^2)} - \alpha)]}$$

and the sign of the latter square root is taken in accordance with the sign of β. Using classical methods (see e.g. [4], [5]) it can be shown that the singular points are

$$\textit{foci} \text{ if } \lambda \neq 0 , \quad \mu \neq 0 ,$$
$$\textit{nodes} \text{ if } \lambda \neq 0 , \quad \mu = 0 ,$$
$$\textit{centres} \text{ if } \lambda = 0 , \quad \mu \neq 0 .$$

In the following we shall introduce conditions under which the trajectorories of (1) behave like the trajectories of the system (2) near $t = \infty$. For this purpose we shall find the detailed integral phase-portrait of the trajectories of (2) exploiting the theory of Riccati differential equation with complex-valued coefficients.

If we define $A(t) = a(t) + ib(t)$, $Z(t) = x(t) + iy(t)$ where $u = x(t)$, $v = y(t)$ is a solution of (1) we have $Z' = u' + iv' = a(t) - u^2 + v^2 + i[b(t) - 2uv]$ so that the function $Z(t)$ is a solution of the equation

$$(4) \qquad Z' = A(t) - Z^2$$

if and only if $Z = u + iv$, u, v being solutions of (1). The Riccati equation corresponding to (2) is of the form

$$(5) \qquad Z' = A - Z^2 ,$$

where $A = \alpha + i\beta$. In what follows let Λ, $-\Lambda$ denote the square roots of A; we shall suppose without loss of generality that $\operatorname{Re} \Lambda \geqq 0$. Further on, R and K denote the set of all real and complex numbers, respectively. If $Z = u + iv \in K$ we denote $\operatorname{Re} Z = u$, $\operatorname{Im} Z = v$, $\bar{Z} = u - iv$, $|Z| = \sqrt{(Z\bar{Z})}$ and $\operatorname{Arg} Z$ the angle Φ such that $\cos \Phi = \operatorname{Re} Z/|Z|$, $\sin \Phi = \operatorname{Im} Z/|Z|$, $0 \leqq \Phi < 2\pi$. A curve $Z = Z(t) = x(t) + iy(t)$ in the argand plane (u, v) is called the trajectory of the equation (4) on an interval i if and only if the function $Z(t)$ satisfies this equation on i. The well known theorem on existence and unicity of solutions quarantees, for any pair $t_1 \in J$, $Z_0 \in K$, the existence of a unique solution Z of (4) defined in a neighborhood of t_1, $Z(t_1) = Z_0$.

In what follows the quite fundamental role is played by the family of circles (Liapunov's functions of (4))

(6) $$\gamma = \frac{A\bar{Z} + \bar{A}Z}{Z\bar{Z} + A\bar{A}}, \quad A \neq 0$$

where γ is a real parameter, $-1 \leqq \gamma \leqq 1$. This equation can be written in the form $|\gamma Z - A| = |A| \sqrt{(1 - \gamma^2)}$ and represents a pencil of circles with limit points A and $-A$ which correspond to the values $\gamma = 1$, $\gamma = -1$, and with the radical axis $A\bar{Z} + \bar{A}Z = 0$ corresponding to $\gamma = 0$. The circle K_γ corresponding to the value γ has the centre A/γ and the radius $r = |A| \sqrt{(1 - \gamma^2)}/|\gamma|$.

Now, we are prepared to prove the following lemma.

Lemma. *The differential equation of the curves which cut all curves* (6) *at the same angle φ is of the form*

(7) $$\frac{dZ}{dt} = \frac{vH}{2A}(A^2 - Z^2),$$

where $H = \sin\varphi + i\cos\varphi$ and $v \neq 0$ is any real constant. The general solution of this equation is

(8) $$Z = A\frac{e^{vHt} - \varkappa}{e^{vHt} + \varkappa}.$$

If $\varkappa = \infty$, then $Z = -A$; the trajectory corresponding to $\varkappa = -1$, namely

(9) $$Z = A\frac{e^{vHt} + 1}{e^{vHt} - 1}$$

passes at $t = 0$ through the point at infinity and has the straight line

(10) $$\mathrm{Im}(\bar{A}HZ) = 0$$

for asymptote. Each trajectory of (7) *starting at the point $Z_0 \neq A(e^{vHt} + 1)/(e^{vHt} - 1)$ is defined for all t. For any trajectory $Z \neq -A\,\mathrm{sgn}(v\,\mathrm{Re}\,H)$ [$Z \neq A\,\mathrm{sgn}(v\,\mathrm{Re}\,H)$] there holds*

(11) $$\lim_{t\to\infty} Z(t) = A\,\mathrm{sgn}(v\,\mathrm{Re}\,H)\,\Big[\lim_{t\to\infty} Z(t) = -A\,\mathrm{sgn}(v\,\mathrm{Re}\,H)\Big].$$

Proof. To find the differential equation of the pencil of circles (6) we differentiate this equation to obtain

$$\frac{(A\bar{Z}' + \bar{A}Z')(Z\bar{Z} + A\bar{A}) - (A\bar{Z} + \bar{A}Z)(Z\bar{Z}' + Z'\bar{Z})}{(Z\bar{Z} + A\bar{A})^2} = 0,$$

so that

$$\bar{A}A^2\bar{Z}' + A\bar{A}^2Z' - A\bar{Z}^2Z' - \bar{A}Z^2\bar{Z}' = 0$$

and

$$\operatorname{Re} AZ'(\bar{A}^2 - \bar{Z}^2) = 0 .$$

Hence the differential equation of (6) is

$$Z' = i\varrho\bar{A}(A^2 - Z^2) , \tag{12}$$

where $\varrho \neq 0$ is a real constant. It is of interest to note that replacing ϱ by any other real constant σ the locus of (12) remains the same. This is the consequence of the fact that the corresponding vectors Z'_ϱ, Z'_σ given by (12) are linearly dependent, so that the above mentioned change of ϱ affects only the velocity of the point moving along the trajectory.

To obtain the differential equation of curves intersecting the pencil (6) at the angle φ it suffices to rotate the vector Z' given by (12) through the angle φ, that means, to multiply the right hand side of the equation (12) by the number $\cos\varphi - i\sin\varphi = -iH$. Putting $v = 2\varrho A\bar{A}$ we obtain the equation (7). This equation can be written in the form

$$\frac{\mathrm{d}Z}{A - Z} + \frac{\mathrm{d}Z}{A + Z} = vH\,\mathrm{d}t$$

if constant solutions

$$Z = \pm A \tag{13}$$

are excluded. Integration of the equation from 0 to t yields

$$\log \varkappa \frac{A + Z}{A - Z} = vHt$$

where

$$\varkappa = \frac{A - Z_0}{A + Z_0}, \quad Z_0 = Z(0) . \tag{14}$$

Hence (8) is the general solution of (7) containing the excluded solutions (13) for $\varkappa = 0$ and $\varkappa = \infty$, too. Note that (14) is a one-to-one correspondence $Z_0 \leftrightarrow \varkappa$; we see that $Z_0 = \infty$ if $\varkappa = -1$.

The solution (8) is defined on the whole real axis if the point Z_0 does not belong to the curve $\varkappa + \mathrm{e}^{vHt} = 0$. Substituting (14) this equation reads

$$Z_0 = -A\frac{\mathrm{e}^{vHt} + 1}{\mathrm{e}^{vHt} - 1} .$$

Replacing t by $-t$ this curve coincides with (9).

Now we have to find the asymptote of (9) Using l'Hospital's rule we get for its slope k

$$k = \lim_{t\to 0} \frac{\operatorname{Im} Z(t)}{\operatorname{Re} Z(t)} = \lim_{t\to 0} \operatorname{Im} \frac{1}{\bar{Z}(t)} \left[\operatorname{Re} \frac{1}{\bar{Z}(t)}\right]^{-1} =$$

$$= \lim_{t\to 0} \operatorname{Im} \frac{\bar{Z}'(t)}{\bar{Z}^2(t)} \left[\operatorname{Re} \frac{\bar{Z}'(t)}{\bar{Z}^2(t)}\right]^{-1} = \lim_{t\to 0} \operatorname{Im} \left\{\frac{\bar{H}}{\bar{\Lambda}} \left[\frac{\bar{\Lambda}^2}{\bar{Z}^2(t)} - 1\right]\right\} .$$

$$\cdot \left(\operatorname{Re} \left\{\frac{\bar{H}}{\bar{\Lambda}} \left[\frac{\bar{\Lambda}^2}{\bar{Z}^2(t)} - 1\right]\right\}\right)^{-1} = \operatorname{Im} \frac{\bar{H}}{\bar{\Lambda}} \left[\operatorname{Re} \frac{\bar{H}}{\bar{\Lambda}}\right]^{-1} = \frac{\operatorname{Im} \Lambda \bar{H}}{\operatorname{Re} \Lambda \bar{H}} .$$

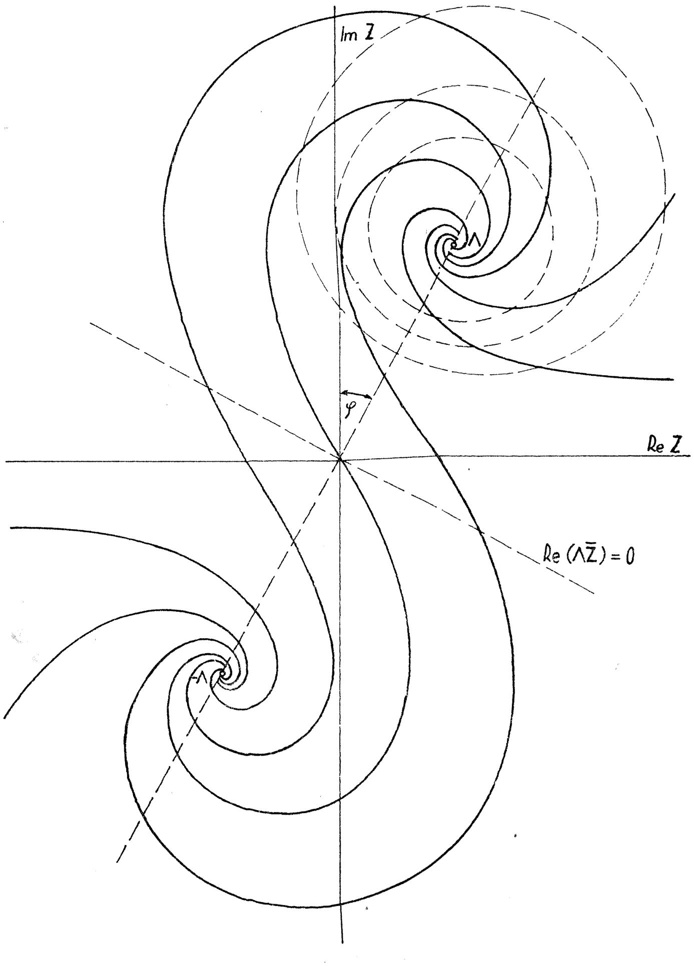

Fig. 1.

The curve (9) being symmetric with respect to the origin and passing through the point $Z_0 = \infty$ only for $t = 0$ has the asymptote containing the origin so that its equation is $\operatorname{Im} Z \operatorname{Re}(\Lambda \bar{H}) = \operatorname{Re} Z \operatorname{Im}(\Lambda \bar{H})$; after rearranging this equation becomes (10). The relations (11) follow immediately from (8).

The proof of the lemma is complete.

For our further considerations the special case $vH = 2\Lambda$ is of great importance and for this reason we introduce it as a theorem.

Theorem 1. *The trajectories of the equation* (5) *cut all the curves of the pencil* (6) *at a constant angle* φ *for which*

$$\sin \varphi = \frac{\operatorname{Re} \Lambda}{|\Lambda|}, \quad \cos \varphi = \frac{\operatorname{Im} \Lambda}{|\Lambda|}.$$

The parametric equations of any trajectory are of the form

$$Z = \Lambda \frac{e^{2\Lambda t} - \varkappa}{e^{2\Lambda t} + \varkappa}$$

where $\varkappa$ *is a suitable constant.*

If $\operatorname{Re} \Lambda > 0$, $\operatorname{Im} \Lambda \neq 0$, *then there is a trajectory having a point at infinity and the real axis for asymptote; all trajectories except two of them,* $Z = \pm\Lambda$, *tend spirally surrounding to the singular point* $\Lambda(-\Lambda)$ *as* $t \to \infty$ $(t \to -\infty)$. *(Fig.* 1.)

If $\operatorname{Re} \Lambda = 0$, *the equation* (12) *coincides for* $\varrho = [\operatorname{Im} \Lambda]^{-1}$ *with* (5) *so that the trajectories of* (5) *form a pencil of circles* (6).

If $\operatorname{Im} \Lambda = 0$, *the trajectories of* (5) *form a pencil of circles*

$$\gamma = i \frac{\Lambda \bar{Z} - \bar{\Lambda} Z}{Z \bar{Z} - \Lambda \bar{\Lambda}}$$

intersecting the circles of (6) *at right angles.*

Theorem 2. *Let* $A(t)$ *be a continuous complex-valued function of the real variable* $t \in J = \langle t_0, \infty)$. *Suppose that there exists a number* $\Lambda \in K$, $\operatorname{Re} \Lambda > 0$ *such that the function*

$$(15) \qquad \Delta(t) = A(t) - \Lambda^2$$

satisfies the condition

$$(16) \qquad \sup_{t \geqq t_0} |\Delta(t)| < |\Lambda| \operatorname{Re} \Lambda .$$

Let $\gamma_0 \in R$, $0 < \gamma_0 < 1$ *be defined by*

(17) $$\sup_{t \geqq t_0} |\Delta(t)| = |A| \operatorname{Re} A \sqrt{(1 - \gamma_0^2)} .$$

If a trajectory $Z(t)$ *of* (4) *satisfies at* $t = t_1 \geqq t_0$ *the inequality*

(18) $$\operatorname{Re} [\bar{A}\, Z(t_1)] > 0 ,$$

then to each γ_1,

(19) $$0 < \gamma_1 < \gamma_0$$

there exists a time $t_2 \geqq t_1$ *so that the trajectory* $Z(t)$ *remains for all* $t > t_2$ *in the interior of the circle* K_{γ_1}

(20) $$\gamma_1 = \frac{A\bar{Z} + \bar{A}Z}{Z\bar{Z} + A\bar{A}} .$$

Proof. The pencil of circles (6) covers the whole plane (u, v). If $Z = Z(t)$ is any trajectory of (4) then the point $Z(t)$ pertains to a circle of (6). This circle corresponds to the value $\gamma(t)$

(21) $$\gamma(t) = \frac{A\bar{Z}(t) + \bar{A}\, Z(t)}{Z(t)\, \bar{Z}(t) + A\bar{A}} .$$

Differentiation yields

$$\gamma'(Z\bar{Z} + A\bar{A})^2 = A^2\bar{A}\bar{Z}' + A\bar{A}^2 Z' - \bar{A}Z^2\bar{Z}' - A\bar{Z}^2 Z' = 2 \operatorname{Re} \{AZ'(\bar{A}^2 - \bar{Z}^2\} .$$

Substituting $Z' = A - Z^2$ and using (15) we get

(22) $$\gamma'(Z\bar{Z} + A\bar{A})^2 = 2 \operatorname{Re} A(A^2 - Z^2)(\bar{A}^2 - \bar{Z}^2) + 2 \operatorname{Re} \{A\Delta(\bar{A}^2 - \bar{Z}^2)\} .$$

Since

$$(A^2 - Z^2)(\bar{A}^2 - \bar{Z}^2) = (Z\bar{Z} + A\bar{A})^2 - (A\bar{Z} + \bar{A}Z)^2 = (Z\bar{Z} + A\bar{A})^2 (1 - \gamma^2) ,$$

the relation (22) can be written in the form

(23) $$\gamma'(t) = 2 \operatorname{Re} A[1 - \gamma^2(t)] + \frac{2 \operatorname{Re} \{A\Delta(t)\, [\bar{A}^2 - \bar{Z}^2(t)]\}}{[Z(t)\, \bar{Z}(t) + A\bar{A}]^2} .$$

Now, the inequality

$$|2 \operatorname{Re} \{A\Delta(\bar{A}^2 - \bar{Z}^2)\}| \leqq 2|A|\, |\Delta|\, |\bar{A}^2 - \bar{Z}^2| = 2|A\Delta|\, [(A^2 - Z^2)(\bar{A}^2 - \bar{Z}^2)]^{1/2} =$$
$$= 2|A\Delta|\, [(Z\bar{Z} + A\bar{A})^2 (1 - \gamma^2)]^{1/2}$$

leads us to the following estimate of the modulus of the second term on the right of (23)

$$\left|\frac{2\ \mathrm{Re}\ \{\Lambda\Delta(\bar{\Lambda}^2 - \bar{Z}^2)\}}{(Z\bar{Z} + \Lambda\bar{\Lambda})^2}\right| \leqq \frac{2|\Lambda\Delta|\ \sqrt{[1-\gamma^2]}}{Z\bar{Z} + \Lambda\bar{\Lambda}} \leqq \frac{2|\Delta|\ \sqrt{[1-\gamma^2]}}{|\Lambda|}.$$

This yields fundamental inequalities

$$(24)\qquad -\frac{2|\Delta(t)|\ \sqrt{[1-\gamma^2(t)]}}{|\Lambda|} \leqq \gamma'(t) - 2\ \mathrm{Re}\ \Lambda[1-\gamma^2(t)] \leqq \frac{2|\Delta(t)|\ \sqrt{[1-\gamma^2(t)]}}{|\Lambda|}.$$

From (16) it is seen that if $\gamma \in R$, $|\gamma| < \gamma_0$ then $|\Delta(t)| < |\Lambda|\ \mathrm{Re}\ \Lambda\ \sqrt{[1-\gamma^2]}$, so that

$$2\ \mathrm{Re}\ \Lambda(1-\gamma^2) > |\Delta(t)|\ \frac{2\sqrt{[1-\gamma^2]}}{|\Lambda|}$$

for $t \geqq t_0$. This inequality and (24) imply that

$$(25)\qquad \gamma'(T) > 0 \quad \text{if} \quad T \in J \quad \text{and} \quad |\gamma(T)| < \gamma_0 .$$

If a point $Z(t_1)$ of the trajectory $Z = Z(t)$ satisfies (18) then there is a circle $K_{\gamma(t_1)}$ of the pencil (6) passing through this point and $\gamma(t_1) > 0$. If $\gamma(t_1) \geqq \gamma_1$ then $\gamma(t) > \gamma_1$ for all $t > t_1$; this is the consequence of the fact that $\gamma(T) = \gamma_1$ implies $\gamma'(T) > 0$ in view of (25). If $\gamma(t_1) < \gamma_1$ we proceed as follows: for all $t \geqq t_1$ for which $\gamma(t) \leqq \gamma_1$ we have in view of (17) and (24)

$$\gamma'(t) \geqq 2\ \mathrm{Re}\ \Lambda(1-\gamma_1^2) - |\Delta(t)|\ \frac{2\sqrt{[1-\gamma_1^2]}}{|\Lambda|} \geqq$$

$$\geqq 2\ \mathrm{Re}\ \Lambda(1-\gamma_1^2) - 2\ \mathrm{Re}\ \Lambda\ \sqrt{[1-\gamma_0^2]}\ \sqrt{[1-\gamma_1^2]} =$$

$$= 2\ \mathrm{Re}\ \Lambda\ \sqrt{[1-\gamma_1^2]}\ (\sqrt{[1-\gamma_1^2]} - \sqrt{[1-\gamma_0^2]}) > 0 .$$

Hence there is a time $t_2 > t_1$ so that $\gamma(t_2) > \gamma_1$. Using the preceding argument we see that $\gamma(t) > \gamma_1$ for $t > t_2$. This means that such a trajectory, after a finite time, will enter the interior of the circle K_{γ_1} and will remain there for $t \to \infty$.

The proof is complete.

Theorem 3. *Assume that $A(t)$ is continuous on J,*

$$(26)\qquad \lim_{t\to\infty} A(t) = \Lambda^2 , \quad \mathrm{Re}\ \Lambda > 0$$

and that the function $\Delta(t)$ defined by (15) satisfies (16). Then every trajectory $Z(t)$ of (4) satisfying (18) at $t_1 \geqq t_0$ tends to the point Λ when $t \to \infty$.

Proof. Let $Z(t)$ be a trajectory of (4) satisfying (18). It is sufficient to prove that if $\gamma_1 \in R$, $0 < \gamma_1 < 1$ then there exists a time $t_2 \geqq t_1$ such that $Z(t)$ remains in the interior of the circle (20) for all $t > t_2$. To this purpose define for $\tau \geqq t_1$ the function $\gamma(\tau) > 0$ by means of the equation

$$\sup_{t \geqq \tau} |\Delta(t)| = |A| \operatorname{Re} A \sqrt{[1 - \gamma^2(\tau)]} .$$

Then (26) implies $\lim_{\tau \to \infty} \gamma(\tau) = 1$. Therefore if $0 < \gamma_1 < 1$ then there is a $t_2 \geqq t_1$ such that $\gamma(t_2) > \gamma_1$. Putting $\gamma_0 = \gamma(t_2)$ it is seen from Theorem 2 that there is a time $t_3 \geqq t_2$ such that $Z(t)$ remains for $t > t_3$ in the circle (20).

Theorem 4. *Let the assumptions of Theorem* 1 *be satisfied. Denote by* $\Phi(Z, t)$ *the angle between the trajectories of the equations* (4) *and* (5) *passing through the point* Z *at a time* t. *Then there holds*

$$\Phi(Z, t) < 90^\circ , \quad \sin \Phi(Z, t) < \frac{\operatorname{Re} A \sqrt{[1 - \gamma_0^2]}}{|A| \sqrt{[1 - \gamma_1^2]}}$$

if Z *belongs to the circle* K_{γ_1} *at the time* $t \geqq t_0$.

Proof. Note that

$$\Phi(Z, t) = \left|\operatorname{Arg}[\Delta(t) - Z^2(t)] - \operatorname{Arg}[A^2 - Z^2(t)]\right| \tag{27}$$

since $\operatorname{Arg}[\Delta(t) - Z^2(t)]$, $\operatorname{Arg}[A^2 - Z^2(t)]$ is the angle between the tangent of the trajectory (4), (5) respectively and the positive real axis. Consequently, there follows from (27)

$$\Phi(Z, t) = \left|\operatorname{Arg} \frac{\Delta(t) - Z^2(t)}{A^2 - Z^2(t)}\right| = \left|\operatorname{Arg} \frac{A^2 + \Delta(t) - Z^2(t)}{A^2 - Z^2(t)}\right| =$$

$$= \left|\operatorname{Arg}\left[1 + \frac{\Delta(t)}{A^2 - Z^2(t)}\right]\right| .$$

Putting

$$R(t) = \left|\frac{\Delta(t)}{A^2 - Z^2(t)}\right|$$

we have (see the proof of Theorem 2)

$$R(t) = \frac{|\Delta(t)|}{\sqrt{[A^2 - Z^2(t)]\,[\bar{A}^2 - \bar{Z}^2(t)]}} = \frac{|\Delta(t)|}{[Z(t)\,\bar{Z}(t) + A\bar{A}]\sqrt{[1 - \gamma^2(t)]}} .$$

Thus at every point of the trajectory $Z(t)$ the following inequality holds

$$R(t) \leqq \frac{|\Delta(t)|}{\Lambda\bar{\Lambda}\sqrt{[1-\gamma^2(t)]}}\,.$$

If the point Z belongs to the circle (20) with γ_1 satisfying (19), it is clear in view of (16) that

$$\frac{|\Delta(t)|}{\Lambda\bar{\Lambda}\sqrt{[1-\gamma_1^2]}} < \frac{\operatorname{Re}\Lambda\sqrt{[1-\gamma_0^2]}}{|\Lambda|\sqrt{[1-\gamma_1^2]}}\,.$$

Thus, $\operatorname{Arg}[1 + \Delta(t)/(\Lambda^2 - Z^2(t))]$ is contained at all points $Z(t)$ of K_{γ_1} between the angles $H(t)$, $-H(t)$ formed by the tangents through the origin to the circle having the centre at $Z = 1$ and the radius $R(t)$, and by the positive real axis. Therefore $\Phi(Z, t) < H(t)$ and the statement follows from the relation $\sin H(t) = R(t)$.

Note. Let $\Psi(Z, t)$ denote the angle between the trajectories of the equations (5) and (12) passing through the point Z at the time t. It holds

$$\Psi(Z, t) = |\operatorname{Arg}\{i\bar{\Lambda}[\Lambda^2 - Z^2(t)]\} - \operatorname{Arg}[\Lambda^2 - Z^2(t)]| = \\ = |\operatorname{Arg}(i\bar{\Lambda})| = |\operatorname{Arg}(\operatorname{Im}\Lambda + i\operatorname{Re}\Lambda)|\,,$$

so that $\sin\Psi(Z, t) = \operatorname{Re}\Lambda/|\Lambda|$ at every point $Z \neq \pm\Lambda$. Comparing this result with the statement of Theorem 4 we see that the tangent of the trajectory $Z(t)$ of the equation (4) is directed into the interior of every circle K_{γ_1} where $0 < \gamma_1 < \gamma_0$.

Theorem 5. *If $A(t)$ is a complex-valued continuous function on J satisfying*

(28) $$\lim_{t\to\infty} A(t) = \Lambda^2\,, \quad \operatorname{Re}\Lambda > 0\,; \quad \int_{t_0}^{\infty} |\mathrm{d}\sqrt{A(t)}| < \infty$$

then there exists a trajectory of the equation (4) defined on the interval $\langle t_1, \infty)$, $t_1 \geqq t_0$ such that

(29) $$\lim_{t\to\infty} Z(t) = \Lambda$$

and

(30) $$\lim_{t\to\infty}\int_{t_0}^{t} [Z(s) - \sqrt{A(s)}]\,\mathrm{d}s$$

exists.

Proof. The existence of a solution satisfying (29) follows from Theorem 3 since the condition (28) implies that there is a time $t_1 \geqq t_0$ such that (16) holds for $t \geqq t_1$.

If $Z(t)$ is a solution defined for $t \geqq t_1$, we have

$$\log [Z(t) + \sqrt{A(t)}] = C + \int_{t_1}^{t} \frac{d[Z(s) + \sqrt{A(s)}]}{Z(s) + \sqrt{A(s)}} =$$

$$= C + \int_{t_1}^{t} \frac{dZ(s)}{Z(s) + \sqrt{A(s)}} + \int_{t_1}^{t} \frac{d\sqrt{A(s)}}{Z(s) + \sqrt{A(s)}}.$$

Hence

$$\int_{t_1}^{t} \frac{dZ(s)}{Z(s) + \sqrt{A(s)}} = \log [Z(t) + \sqrt{A(t)}] - \int_{t_1}^{t} \frac{d\sqrt{A(s)}}{Z(s) + \sqrt{A(s)}} - C$$

and in view of (29) and (28) $Z(t)$ satisfies the condition

$$\lim_{t \to \infty} \int_{t_1}^{t} \frac{dZ(s)}{Z(s) + \sqrt{A(s)}} \tag{31}$$

exists. The differential equation (4) yields $Z'(s)/(Z(s) + \sqrt{A(s)}) = Z(s) - \sqrt{A(s)}$ so that (30) follows from (31).

Theorem 6. *Assume that $A(t)$ is continuous on J. Let there exist a constant $\Lambda \in K$, $\Lambda \neq 0$ such that the function* (15) *satisfies the condition*

$$\int_{t_0}^{\infty} |\Delta(t)| \, dt < \infty . \tag{32}$$

Then every solution of (4) *defined for $t \to \infty$ satisfies either*

$$\lim_{t \to \infty} Z(t) = \Lambda, \quad \lim_{t \to \infty} \int^{t} |Z(s) - \Lambda| \, ds \tag{33}$$

exists, or

$$\lim_{t \to \infty} Z(t) = -\Lambda, \quad \lim_{t \to \infty} \int^{t} |Z(s) + \Lambda| \, ds \tag{34}$$

exists.

Proof. Let $Z = Z(t)$ be a trajectory of (4) defined for $t \geqq t_1$. Then the point $Z(t)$ belongs to a circle of the pencil (6) which corresponds to the value $\gamma(t)$ satisfying the equation (21). By (24) and the fact that

$$|\gamma(t)| \leqq 1 \tag{35}$$

there follows

$$-\frac{|\Delta(t)|}{|\Lambda| \operatorname{Re} \Lambda} \leqq \frac{\gamma'(t)}{2 \operatorname{Re} \Lambda} - [1 - \gamma^2(t)] \leqq \frac{|\Delta(t)|}{|\Lambda| \operatorname{Re} \Lambda}. \tag{36}$$

Integrating these inequalities over $\langle t_1, t\rangle$ we see in view of (32) that the hypothesis $\int_{t_1}^{\infty} [1 - \gamma^2(t)]\, dt = \infty$ implies $\gamma(t)/\text{Re}\,\Lambda \to \infty$ when $t \to \infty$; but this contradicts (35) and therefore

$$(37) \qquad \int_{t_1}^{\infty} [1 - \gamma^2(t)]\, dt < \infty\,.$$

Moreover, from (24) we obtain the inequality

$$|\gamma'(t)| \leqq 2\,\text{Re}\,\Lambda[1 - \gamma^2(t)] + \frac{2|\Delta(t)|}{|\Lambda|}$$

for $t \geqq t_1$. From here and from (37) we have

$$(38) \qquad \int_{t_1}^{\infty} |\gamma'(t)|\, dt < \infty\,.$$

Therefore $\gamma(t)$ converges when $t \to \infty$ and with respect to (37) it holds either

$$(39) \qquad \lim_{t\to\infty} \gamma(t) = 1\,, \quad \text{or} \quad \lim_{t\to\infty} \gamma(t) = -1\,.$$

This means that

$$(40) \qquad \lim_{t\to\infty} Z(t) = \Lambda\,,$$

or

$$(41) \qquad \lim_{t\to\infty} Z(t) = -\Lambda\,.$$

Now, let us consider the function

$$(42) \qquad c(t) = \frac{|Z(t) \mp \Lambda|^2}{\Lambda\,\bar{Z}(t) + \bar{\Lambda}\,Z(t)}\,,$$

where the sign $-$ $[+]$ is chosen if (40) [(41)] holds. By simple induction we have

$$c(t) = \frac{1 \mp \gamma(t)}{\gamma(t)}\,.$$

From here it is seen that $c(t)$ is defined for large t since $\gamma(t) \neq 0$ near $t = \infty$ in view of (39). By (40), (41) and (42) we have

$$(43) \qquad |Z(t) \mp \Lambda| \leqq K\left[\frac{1 \mp \gamma(t)}{|\gamma(t)|}\right]^{1/2} = K\left[\frac{1 - \gamma^2(t)}{|\gamma(t)|\,\sqrt{[1 \pm \gamma^2(t)]}}\right]^{1/2}$$

where K is a suitable constant. On the other hand it follows from (24)

$$-\frac{|\Delta(t)|}{|\Lambda| \operatorname{Re} \Lambda} \leqq \frac{-\gamma(t)\,\gamma'(t)}{2 \operatorname{Re} \Lambda \sqrt{[1-\gamma^2(t)]}} - \gamma(t)\,[1-\gamma^2(t)]^{1/2} \leqq \frac{|\Delta(t)|}{|\Lambda| \operatorname{Re} \Lambda}.$$

Using the same argument as in the proof of (37) we get

$$\left| \int^{\infty} \gamma(t)\,[1-\gamma^2(t)]\, \mathrm{d}t \right| < \infty .$$

Evidently, this condition is equivalent to the following one

$$(44) \qquad \int^{\infty} [1-\gamma^2(t)]\, \mathrm{d}t < \infty$$

since $\gamma^2(t) \to 1$ when $t \to \infty$. Since

$$\sqrt{\frac{1-\gamma^2(t)}{|\gamma(t)|\,[1 \pm \gamma(t)]}} \leqq L\sqrt{[1-\gamma^2(t)]}$$

for a suitable constant L and large t, it is seen from (41) that

$$|Z(t) \mp \Lambda| \leqq L\sqrt{[1-\gamma^2(t)]}\,.$$

This inequality and (44) quarantee the convergence of the integrals in (33), (34).

The proof is complete.

We will complete our considerations by noting that the preceding theorems have their analogies for the real system (1). We will not introduce them here since they may be stated without difficulties.

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