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BIREGULAR SEMIGROUPS I

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In [1], R. F. ARENS and I. KAPLANSKY introduced the concept of biregularity for rings. The present paper is devoted to the study of the analogous concept for semigroups.

It will be shown that biregularity generalizes the concept of an inverse semigroup which is a union of groups, characterized by A. H. CLIFFORD in [3]. If each $\mathcal{J}$-class is a subsemigroup, then in the presence of minimality conditions or of Croisot's regularity conditions, these two classes of semigroups coincide and are, moreover, the class of biregular semigroups satisfying a natural uniqueness condition.

With each biregular semigroup S a groupoid S^* is associated, where S^* is a semilattice union of 0-simple semigroups and satisfies a categorical condition. If S is an inverse semigroup which is a union of groups, then S^* is Clifford's construction which characterizes such semigroups. Also, it is shown that every $\mathcal{J}$-class of a biregular semigroup is partially isomorphic to a 0-simple semigroup.

1. PRELIMINARIES

A semigroup S is said to be *biregular* if each principal two-sided ideal of S is generated by an idempotent in the center of S. A semigroup S is *bi-inverse* if each principal two-sided ideal of S is generated by a unique idempotent of S and this idempotent is in the center of S.

Note that if in a semigroup S we have $S'e_1S' = S'e_2S'$, where e_1 and e_2 are idempotents in the center of S, then we must have $e_1 = e_2$. Thus, a semigroup S which is biregular and not bi-inverse, contains at least one element $a \in S$ such that $S'aS'$ has an idempotent generator which is not in the center of S.

For example: If $S = C(p, q)$ is the bicyclic semigroup then, since S is simple with identity; S is biregular. However, since the only idempotent in the center of S is the identity; S is not bi-inverse.

Throughout this paper we shall use "ideal" for two-sided ideal. The standard terminology and notation used is that of [4].

If S is a semigroup, the center of S will be denoted by $Z(S)$; the collection of idempotents of S will be denoted by $E(S)$ and $EZ(S)$ will stand for $E(Z(S))$.

We can first note that a simple semigroup with an identity is biregular and its center is both biregular and simple and since it is commutative is a group. Also, the concepts of biregularity and regularity are independent. Indeed, let $T(X)$ be the semigroup of all transformations of a set X into itself, then as is well known $T(X)$ is regular and if $|X| > 2$, the only idempotent in the center of $T(X)$ is the identity. Since $T(X)$ is not simple, $T(X)$ is not biregular. Also, if $C(S)$ denotes Bruck's semigroup (see [2]) where S is not a regular semigroup, then $C(S)$ is not regular; however, it is biregular since it is simple with an identity.

In [7], D. R. Morrison considered some properties of biregular rings; most of his results can easily be adapted to biregular semigroups. We will use one of his results, namely;

Lemma 1.1. *If S is a biregular semigroup then $Z(S)$ is biregular.*

The proof is omitted since it is exactly Morrison's proof.

In the sequel we shall use the following notation: If A is a subset of B then $B - A$ will denote the set-theoretic complement of A in B. If A is an ideal of the semigroup B, then B/A will mean the Rees' quotient of B by A.

Lemma 1.2. *If S is a biregular semigroup and e is an element of $EZ(S)$ then $J(e) - L_e$ $(J(e) - R_e)$ is a left (right) ideal of S.*

Proof. Let a be in $J(e) - L_e$ and x an arbitrary element of S. Clearly xa is in $J(e)$. Suppose that xa is in L_e; then $Sxa = Se$, thus $e = yxa$ for some y in S; hence $Se \subseteq Sa$. Since e is in $EZ(S)$, $Se = J(e)$ so that $Se \subseteq Sa \subseteq J(e) = Se$, or a is in L_e, a contradiction. Similarly for $J(e) - R_e$.

Lemma 1.3. *If S is a biregular semigroup then $Z(S)$ is a commutative inverse semigroup which is a union of groups.*

Proof. By lemma 1.1, if S is biregular then so is $Z(S)$, thus, $Z(S)\, a\, Z(S) = Z(S)\, a = a\, Z(S)$ for all a in $Z(S)$. Hence, every one-sided principal ideal of $Z(S)$ is generated by an idempotent, and since $Z(S)$ is commutative, it is an inverse semigroup. Since, $\mathscr{J} = \mathscr{H}$ and every $\mathscr{J}$-class has an idempotent, it is a union of groups.

2. BI-INVERSE SEMIGROUPS

The class of bi-inverse semigroups can now be characterized with the aid of one more preliminary result.

Lemma 2.1. *Let S be a bi-inverse semigroup, then S is an inverse semigroup if and only if $\mathcal{J} = \mathcal{D}$ and in this case S is a union of groups.*

Proof. Suppose S is bi-inverse and $\mathcal{J} = \mathcal{D}$. Then every $\mathcal{D}$-class is regular. Moreover every idempotent of S is in $E\,Z(S)$, thus S is inverse.

Conversely, suppose that S is bi-inverse and inverse. Since each $\mathcal{J}$-class contains a unique idempotent and S is regular, we must have $\mathcal{J} = \mathcal{D}$. In this case, we further have that $\mathcal{D} = \mathcal{J} = \mathcal{H}$; and so S is a union of groups.

Theorem 2.1. *A semigroup S is bi-inverse if and only if S is an inverse semigroup which is a union of groups.*

Proof. Let e be in $E\,Z(S)$. Since D_e is a regular $\mathcal{D}$-class of S, with a unique idempotent, then $D_e = L_e = R_e = H_e$. Let $S_e = J(e) - H_e = (J(e) - L_e) \cap (J(e) - R_e)$. Then, by lemma 1.2, S_e is an ideal of S and it contains $I(e) = J(e) - J_e$. Thus, since $I(e)$ is maximal, $S_e = I(e)$ and hence $J_e = H_e$ and so S is an inverse semigroup which is a union of groups.

Conversely, it is readily verifiable that every inverse semigroup which is a union of groups is bi-inverse.

3. BIREGULAR SEMIGROUPS AND MINIMAL CONDITIONS

A semigroup S is said to satisfy $M_L^*(M_R^*)$ if the set of all $\mathcal{L}-(\mathcal{R}-)$ classes of S contained in a $\mathcal{J}$-class of S, contains a minimal member, with respect to the usual ordering of classes.

A semigroup S is said to be *right* (*left, intra-*) *regular* if for all a in S, (a, a^2) is in $R(L, J)$.

Theorem 3.1. *The following conditions are equivalent for a biregular semigroup in which each $\mathcal{J}$-class is a subsemigroup of S.*

(1) *S satisfies M_L^* or M_R^*.*

(2) *S is right and left regular.*

(3) *S is bi-inverse.*

Proof. Since each $\mathcal{J}$-class of S is a subsemigroup, it is a simple semigroup and if it satisfies M_L^* then by [8], theorem 2.5, each $\mathcal{J}$-class has a primitive idempotent, and is thus a completely simple semigroup, since moreover each $\mathcal{J}$-class has an identity, it is a group. Therefore, S is bi-inverse. Conversely an inverse semigroup which is a union of groups clearly satisfies M_L^*. Proving the equivalence of (1) and (3). Now, if S is right and left regular then by [5], S is a union of groups and thus in particular each $\mathcal{J}$-class of S is a simple semigroup which is a union of groups and hence completely simple and thus a group. Conversely, if S is bi-inverse then since $\mathscr{L} = \mathscr{R} = \mathscr{H}$ S is both left and right regular. Proving the equivalence of (2) and (3).

4. BIREGULAR $\mathscr{J}$-CLASSES

Let S be a biregular semigroup. For each a in S, denote by $e(a)$ the unique central idempotent in J_a. If e is in $E\,Z(S)$, let $T_e = \{x \text{ in } S : e \leqq e(x)\}$ and let $S_e = S - T_e$.

Let $a \in S_e$ and $b \in S$, then if $ab \in T_e$ we would have $a(ab) \geqq e$; but $e(a)\,e(b) \geqq e(ab)$ therefore, $e(a)\,e(b) \geqq e$ and $e(a) \geqq e$ a contradiction. Thus, S_e is an ideal of S. Let $S_e^* = S/S_e$. If $J^*(e)$ and J_e^* are respectively the principal ideal generated by e and the $\mathscr{J}$-class of e, in $S_{e_1}^*$ then using lemma 2.1 of [8], the following equalities follow easily: $J_e^* = J_e \cup \{0\} = (J(e) \cap T_e) \cup \{0\} = J^*(e)$. Thus, J_e^* is a minimal 0-simple ideal of S_e^*.

Let Y be a semilattice isomorphic to $E\,Z(S)$ where α corresponds to e_α. To each α in Y let 0_α denote the zero of $S_e^* = S_\alpha$. If α and β are in Y and such that $\alpha \geqq \beta$ define a mapping $\varphi_{\alpha,\beta}$ from $J_{e_\alpha}^* = J_\alpha^*$ to $J_{e_\beta}^* = J_\beta^*$ by

$$a\varphi_{\alpha,\beta} = \begin{cases} ae_\beta & \text{if} \quad a_\alpha \neq 0_\alpha \\ 0_\beta & \text{if} \quad a_\alpha = 0_\alpha \,. \end{cases}$$

Then, $\varphi_{\alpha,\beta}$ is a *partial homomorphism*, i.e. (i) $0_\alpha\varphi_{\alpha,\beta} = 0_\beta$ and (ii) $(ab)\,\varphi_{\alpha,\beta} = (a\varphi_{\alpha,\beta})\,(b\varphi_{\alpha,\beta})$ for all a and b such that $ab \neq 0_\alpha$. The following is readily verifiable:

(1) $a \neq 0_\alpha$ implies that $ae_\beta \neq 0_\beta$.

(2) $\varphi_{\alpha,\alpha}$ is the identity on J^*.

(3) if $\alpha \geqq \beta \geqq \gamma$ then $\varphi_{\alpha,\gamma} = \varphi_{\alpha,\beta}\varphi_{\beta,\gamma}$.

Let $S^* = \bigcup\{J_\alpha^* : \alpha \text{ is in } Y\}$ and define an operation on S^* by

$$a_\alpha \,.\, b_\beta = (a_\alpha\varphi_{\alpha,\alpha\beta})\,(b_\beta\varphi_{\beta,\alpha\beta}) \text{ for } a_\alpha \text{ in } J_\alpha^* \text{ and } b_\beta \text{ in } J_\beta^* \,.$$

Then, . is not necessarily associative, but satisfies the following categorical condition:

(C) If $a_\alpha \,.\, b_\alpha \neq 0_{\alpha\beta}$ and $b_\beta \,.\, c_\gamma \neq 0_{\beta\alpha}$ then $(a_\alpha \,.\, b_\beta) \,.\, c_\gamma = a_\alpha \,.\, (b_\beta \,.\, c_\gamma)$.

Furthermore, the mapping i form $J_e^* - \{0_\alpha\}$ to J_e that sends a (in $S_{e_\alpha}^*$) to a (in S) is a partial-isimorphism.

We can thus state:

Theorem 4.1. *Let S be a biregular semigroup. Then, there is a groupoid S^* which is a semilattice union of 0-simple semigroups with identity, J_α^*, such that:*

(1) *S^* satisfies condition (C).*

(2) *Each $\mathscr{J}$-class of S is partial-isomorphic to a J_α^*.*

(3) *The semilattice of S^* is isomorphic to $E\,Z(S)$.*

(4) *A $\mathcal{J}$-class J_e of S is a subsemigroup of S if and only if J_e is isomorphic to J_e^*.*

(5) *If S is bi-inverse, then S^* is Clifford's description of inverse semigroups which are unions of groups.*

References

[1] *R. F. Arens* and *I. Kaplansky:* Topological representation of algebras, Trans. Amer. Math. Soc. *63* (1948), 457—481.

[2] *R. H. Bruck:* A survey of binary systems, Ergebnisse der Math. Heft *20,* Springer, Berlin, 1958, 185.

[3] *A. H. Clifford:* Semigroups admitting relative inverses, Annals of Math. *42* (1941), 1037—1048.

[4] *A. H. Clifford* and *G. B. Preston:* Algebraic theory of semigroups, vol. 1, Amer. Math. Soc. Surveys *7,* 1961.

[5] *R. Croisot:* Demi-groupes inversifs et demi-groupes reunions de demi-groupes simples, Ann. Sci. Ecole Norm. Sup. (3) *70* (1953), 361—379.

[6] *D. D. Miller* and *A. H. Clifford:* Regular *D*-classes in semigroups, Trans. Amer. Math. Soc. *82* (1956), 270—280.

[7] *D. R. Morrison:* Biregular rings and the ideal lattice isomorphisms, Proc. Amer. Math. Soc. *6* (1952), 46—49.

[8] *W. D. Munn:* Semigroups satisfying minimal conditions, Proc. Glasgow Math. Assoc. *3* (1957), 145—152.

[9] *M. Petrich:* Semigroups certain of whose subsemigroups have identities, Czechoslovak Math. J. *16* (*91*), (1966), 186—198.

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