Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 20 (1970), No. 4, 599–602

Persistent URL: http://dml.cz/dmlcz/100985

Czechoslovak Mathematical Journal, 20 (95) 1970, Praha

SOME REMARKS CONCERNING STABLE ATTRACTORS*)

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(Received July 7, 1969)

Introduction. In [1] the following theorem is established: Let (X, π) be a dynamical system on the locally compact metric space X. If $M \subset X$ be compact, invariant, and (positively) asymptotically stable, then there is a real valued continuous mapping, $v : A(M) \to R^+$, from the region of attraction of M into the non-negative reals which is uniformly unbounded on $A(M)$ and, in addition, satisfies:

(†) $v(x) = 0$ iff $x \in M$;

(‡) $v(\pi(x, t)) \equiv e^{-t} . v(x)$ for every $(x,t) \in A(M) \times R$.

We wish to establish some consequences of this theorem in the basic notation of [2]. All attracting is positively.

Definition 1. Let (X, π) be a dynamical system and let $M \subset X$ be attracting. A subset $N \subset X$ is called *pre-admissible* for M if N is a neighborhood of M in $A(M.)$ A subset $N \subset X$ is called *admissible* for M if N is pre-admissible for M and N is positively invariant.

Note that $A(M)$ is always admissible for M and if N is pre-admissible for M, then $\gamma^+(N) = N^*$ is admissible for M.

Theorem 1. *Let (X, π) be a dynamical system on a locally compact metric space and let $M \subset X$ be compact, invariant, and asymptotically stable. If $N \subset X$ is admissible for M, then there is a compact subset $N' \subset N$ which is admissible for M and which is a (strong) deformation retraction of N.*

Proof. Let W be open in X with $M \subset W \subset \overline{W} \subset N$ with $\overline{W}$ compact. Let $v : A(M) \to R^+$ be as above. We wish to select $k > 0$ so that $v^{-1}[0, k] \subset W$. Suppose no such k exists. Since v is uniformly unbounded on $A(M)$, there is a compact

*) The author was partially supported by the National Science Foundation under Grant No. NSF-GE-7938.

subset $K_1 \nsubseteq A(M)$ such that $v(x) \geqq 1$ for every $x \in A(M) \setminus K_1$. Then for each integer $n = 2, 3, \ldots$ there is an $x_n \in v^{-1}[0, 1/n] \cap (X \setminus W)$ and, hence, a subsequence converging to $x_0 \in K_1 \cap (X \setminus W)$. Since v is continuous on $A(M)$, we must have $v(x_0) = 0$ which contradicts (†). Therefore such a $k > 0$ exists.

Now let $x \in N \setminus v^{-1}[0, k)$ and for each real t define $\pi_x(t) \equiv \pi(x, t)$. Then the composite mapping $v \circ (\pi_x \mid R^+) : R^+ \to R^+$ is continuous with $[v \circ (\pi_x \mid R^+)](0) \geqq \geqq k$ and $[v \circ (\pi_x \mid R^+)](t) \to 0$ as $t \to \infty$. Hence, there is a $t_x \in R^+$ such that $[v \circ (\pi_x \mid R^+)](t_x) = k$ and t_x is unique by (‡).

Define a mapping $p : N \setminus v^{-1}[0, k) \to R^+$ by $p(x) = t_x$. The uniqueness of t_x insures the continuity of p.

Finally, define a mapping $H : N \times [0, 1] \to N$ by

$$H(y, s) = \begin{cases} y, & (y, s) \in v^{-1}[0, k] \times [0, 1] \\ \pi(y, s \,.\, p(y)), & (y, s) \in (N \setminus v^{-1}[0, k)) \times [0, 1] \,. \end{cases}$$

Then $H(y, 0) = y$ for every $y \in N$, $H(y, s) = y$ for every $(y, s) \in v^{-1}[0, k] \times [0, 1]$, $H(N \times \{1\}) = v^{-1}[0, k]$, and H is continuous. Hence, $v^{-1}[0, k]$ is a (strong) deformation retraction of N. Put $N' = v^{-1}[0, k]$ and the proof is complete.

Theorem 2. *Under the same hypothesis as Theorem* 1, *if $N \subset X$ is preadmissible for M, then there is a compact subset $N' \subset N$ which is admissible for M and which is a retraction of N.*

Proof. Let $W \subset X$ and $k > 0$ be chosen as in Theorem 1. Define a mapping $r : N \to v^{-1}[0, k]$ by

$$r(x) = \begin{cases} \pi(x, p(x)), & x \in N \setminus v^{-1}[0, k) \\ x, & x \in v^{-1}[0, k] \end{cases}$$

where $p : N \setminus v^{-1}[0, k) \to R^+$ is the mapping constructed in the proof of Theorem 1. Put $N' = v^{-1}[0, k]$. Then r is continuous, r is the identity mapping on N', N' is positively invariant by (‡), and $v^{-1}[0, k) \subset N'$. Hence, N' is admissible and the proof is complete.

If X is a compact (Hausdorff) space, G an $\{R$-module, compact abelian group$\}$, G' an R-module, and r an integer $\geqq 0$, then $\{H_r(X), H^r(X)\}$ denotes the r-dimensional Čech $\{$homology, cohomology$\}$ group of X over $\{G, G'\}$. Also $\{\tilde{H}_0(X), \tilde{H}^0(X)\}$ denotes the 0-dimensional augmented Čech $\{$homology, cohomology$\}$ group of X over $\{G, G'\}$. If G is a vector space over a field, then we denote the dimension of $H_r(X)$ by $\beta_r(X)$.

Theorem 3. *Let (X, π) be a dynamical system on a locally compact metric space. Let $M \subset X$ be compact, invariant, and asymptotically stable. If $N \subset X$ is compact* *and admissible for M, then, for every integer $r \geqq 0$, $H_r(M) \simeq H_r(N)$ and $H^r(M) \simeq \simeq H^r(N)$.*

Proof. Choose $N' \subset N$ such that $N' = v^{-1}[0, k]$, $k > 0$, and N' is a (strong) deformation retraction of N. Then for $r \geqq 0$, $H_r(N') \simeq H_r(N)$ and $H^r(N') \simeq \simeq H^r(N)$. It suffices to show $H_r(M) \simeq H_r(N')$ and $H^r(M) \simeq H^r(N')$. Let $\mathscr{V} = \{V_n = v^{-1}[0, k/n] \mid n \geqq 1\}$. Then $\{\mathscr{V}, f\}$ is an inverse sequence of compact spaces over the positive integers, Z^+, where $f_{nm} : V_n \to V_m$ is an inclusion mapping whenever $m \leqq n$ in Z^+. Then V_∞ is homeomorphic with $V' = \bigcap_{k=1}^{\infty} V_k = M$. Hence, $H_r(M) \simeq H_r(V_\infty)$ and $H^r(M) \simeq H^r(V_\infty)$.

Now if $m \in Z^+$, V_{m+1} is a (strong) deformation retraction of V_m. To see this, note that for each $x \in V_m \setminus v^{-1}[0, k/m + 1)$ there is a unique $t_x \geqq 0$ such that $v(\pi(x, t_x)) = k/m + 1$. The mapping $p : V_m \setminus v^{-1}[0, k/m + 1) \to R^+$ determined by $p(x) = t_x$ is continuous and we may define $H : V_m \times [0, 1] \to V_m$ by

$$H(x, s) = \begin{cases} x, & (x, s) \in v^{-1}[0, k/m + 1] \times [0, 1] \\ \pi(x, s \,.\, p(x)), & (x, s) \in (V_m \setminus v^{-1}[0, k/m + 1)) \times [0, 1] \end{cases}$$

to obtain the desired deformation.

Hence, whenever $m \leqq n$ in Z^+, $f_{nm} : V_n \to V_m$ induces isomorphisms onto $f_{nm*} : H_r(V_n) \to H_r(V_m)$ and $f_{nm**} : H^r(V_m) \to H^r(V_n)$. Therefore, $\lim\limits_{\rightarrow} \{H^r(.), f_{**}\} \simeq H^r(V_1) = H^r(N')$ and $\lim\limits_{\leftarrow} \{H_r(.), f_*\} \simeq H_r(V_1) = H_r(N')$. But by the continuity axiom, $\lim\limits_{\rightarrow} \{H^r(.), f_{**}\} \simeq H^r(V_\infty)$ and $\lim\limits_{\leftarrow} \{H_r(.), f_*\} \simeq H_r(V_\infty)$. Hence, $H_r(M) \simeq H_r(N')$ and $H^r(M) \simeq H^r(N')$ and the proof is complete.

Corollary. *Under the hypothesis of the theorem, if $N \subset X$ is preadmissible for M and contractible, then $H_r(M) = 0$ for $r \neq 0$, $H^r(M) = 0$ for $r \neq 0$, $\tilde{H}_0(M) = 0$ and $\tilde{H}^0(M) = 0$.*

Proof. By Theorem 2 there is a compact admissible $N' \subset N$ with is a retraction of N. Hence, N is contractible. Hence, $H_r(N') = 0 = H^r(N')$ for $r \neq 0$ and $\tilde{H}_0(N') = 0 = \tilde{H}^0(N')$. By Theorem 3 these are the same for M.

Corollary. *Under the hypothesis of the theorem, M is connected if and only if $A(M)$ is connected.*

Proof. Suppose $A(M)$ is connected. Then there is an $N \subset A(M)$ which is compact, admissible for M, and a retraction of $A(M)$. Hence, $\tilde{\beta}_0(N) = 0$. Therefore, $\tilde{\beta}_0(M) = 0$ and M is connected. The rest of the proof follows directly.

Note that if $X = R^{n+1}$, $n \geqq 1$, and $A(M) = R^{n+1}$, then M and $R^{n+1} \setminus M$ are connected. For M disconnects R^{n+1} if and only if $H_n(M; R_1) \neq 0$ where R_1 is the compact abelian group of real numbers modulo 1.

Finally, under the hypothesis of Theorem 3, there is a continuous mapping of $A(M)$ into the non-negative real number, $v : A(M) \to R^+$, such that M is the zero-set of v and each level set of v is the same homotopy type as $A(M) \setminus M$.

References

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