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A DUALITY THEOREM FOR STANDARD THREADS*)

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I. INTRODUCTION

The characters of a topological groups are continuous complex-valued homomorphisms for the simple reason that no continuous non-trivial real-valued characters exist. In the case of a topological semigroup the situation is very different since there are semigroups for which no continuous non-trivial semicharacters exist. One approach is to consider equivalence classes of measurable complex-valued semicharacters as was done in [1]. In this paper we show that by considering such equivalence classes for real-valued semicharacters a Pontrjagin type of duality theorem can be obtained for a certain class of semigroups.

A *standard thread* [2] is a compact semigroup S with a total order such that:

- i) the order topology is the given topology;
- ii) S is connected in the order topology;
- iii) S has a maximal element and it is an identity;
- iv) S has a minimal element and it is a zero.

A *nil thread* [2] is a standard thread having no interior idempotent but at least one non-zero nilpotent element. A *unit thread* [2] is a standard thread with no interior idempotent element and no nilpotent element. With any thread S we associate its order dual S^0 obtained from S by inverting the order relation.

Suppose E is a compact totally ordered set. We define two distinct points of E to be adjacent if there is no element of E between them. The lesser of two adjacent elements will be called an initial element; the greater a terminal element. An

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element which is not an initial element will be called a limit element. With each initial element e of E we associate a nil or unit thread S_e . Identify e with the zero element of S_e and let S'_e denote S_e with its identity deleted. If e is a limit point, define $S_e = S'_e =$ one point semigroup $\{e\}$. The *ordinal sum* of the semigroups $\{S'_e\}_{e \in E}$ is denoted by $\sum_e S'_e$ and is the semigroup $S = \bigcup_{e \in E} S'_e$ where the product $a \circ b$ of two elements in S is given by

$$\begin{aligned} a \circ b &= \text{usual product in } S'_e \text{ if } a, b \in S'_e = \\ &= ab = ba = a \text{ if } a \in S'_e, b \in S'_f \text{ and } e < f. \end{aligned}$$

The canonical representations in the following theorems are due to CLIFFORD [2] and form the basis for our work.

Theorem. *Every standard thread S is the ordinal sum of a compact totally ordered set of half open nil threads, half open unit threads, and one element semigroups.*

Theorem. *Every unit thread is isomorphic to $[0, 1]$ with ordinary multiplication, and every nil thread N is isomorphic to $[\frac{1}{2}, 1]$ with $x \circ y = \max\{\frac{1}{2}, xy\}$ for $x, y \in [\frac{1}{2}, 1]$.*

II. THE REAL DUAL OF A STANDARD THREAD

In order to investigate the nature of the real-valued semicharacters of a standard thread it suffices to examine the duals of the canonical elements.

Proposition 1. *Let S be the semigroup $(0, 1)$ under ordinary multiplication and let S^* denote the semigroup of bounded real-valued non-constant semicharacters on S .*

Then S^ is isomorphic to the order dual of the semi-group of positive real numbers under addition.*

Proof. The proof of this proposition, although very simple, is rather long and is broken down into six steps.

i) Consider the possible zeros for an element $\tau \in S^*$:

If $\tau(x) = 0$, then clearly $\tau(y) = 0$ for all $y < x$. Moreover, if $\tau \not\equiv 0$, then there exists a z such that $\tau(z) \neq 0$ but $z < 1$ implies that for some n we have $z^n < x$ so $\tau(z^n) = 0 = \tau(z)^n$. Hence $\tau \equiv 0$ if it is zero at any point.

ii) Every element of S^* is order preserving:

Suppose $x < y$. Then $x = sy$ for some $s \in S$ hence $\tau(x) = \tau(sy) = \tau(s) \cdot \tau(y)$ but $\tau(s) \leq 1$ and thus $\tau(x) \leq \tau(y)$.

- iii) $\tau \in S^*$ implies $\tau(y^r) = (\tau(y))^r$ for any rational $r > 0$:
- iv) $\tau \in S^*$ implies τ is continuous in the order topology:

As result of (ii), τ is monotone and hence has only jump discontinuities. In particular, suppose

$$\tau^-(y) = \sup \{ \tau(x) : x < y \} < \tau(y)$$

for some y . Pick an $x_0 < y$ and choose a rational r such that

$$\tau^-(y) < [\tau(x_0)]^r < \tau(y).$$

Then, from (iii), we have $\tau^-(y) < \tau(x_0^r) < \tau(y)$ which is a contradiction. A similar argument is used for $\tau^+(y) = \inf \{ \tau(x) : x > y \}$.

- v) If $\tau \in S^*$, then $\tau(x) = x^{\alpha_0}$ for some $\alpha_0 \in (0, \infty)$:

It is clear that $\tau(x) = 1$ for some x implies $\tau \equiv 1$. If $\tau \not\equiv 1$, then $\tau(\frac{1}{2}) = (\frac{1}{2})^{\alpha_0}$ for some real number α_0 and it follows from (iii) and (iv) that $\tau(x) = x^{\alpha_0}$ for all $x \in (0, 1)$.

- vi) S^* is isomorphic to $\{R^+, +\}^0$:

Use the obvious mapping $\Phi : S^* \rightarrow R^+$ by $\Phi : x^\alpha \rightarrow \alpha$.

Corollary 2. *If S is the semigroup $(0, 1)$ under ordinary multiplication and S^* is the semigroup of bounded real-valued non-constant semicharacters on S , then S^{**} is isomorphic to S .*

Proof. Define the mapping $\psi : (R^+)^0 \rightarrow (0, 1)$ by $\psi : \alpha \rightarrow 1/e^\alpha$.

Corollary 3. *If T is a unit thread, m is Lebesgue measure on T , and T^* is the semigroup of equivalence classes of bounded, measurable, real-valued semicharacters on T , then T^* is isomorphic to the order dual of $[0, \infty]$ under addition and T^{**} is isomorphic to T .*

Proof. We let $[\theta]$ denote the equivalence class containing the two semicharacters τ_θ and τ'_θ where

$$\begin{aligned} \tau_\theta(x) \equiv 0 \quad \tau'_\theta(x) = 0 & \text{ if } 0 \leq x < 1 \\ & = 1 \quad \text{if } x = 1 \end{aligned}$$

and similarly $[1]$ denotes the equivalence class containing τ_1 and τ'_1 where

$$\begin{aligned} \tau_1(x) \equiv 1 \quad \tau'_1(x) = 0 & \text{ if } x = 0 \\ & = 1 \quad \text{if } 0 < x \leq 1. \end{aligned}$$

Since any semicharacter has the value 0 or 1 at an idempotent and any non-constant semicharacter must have values 0 at zero and 1 at the identity it follows

that $T^* = S^* \cup \{[\theta]\} \cup \{[1]\}$. We can now map T^* directly onto $[0, 1]$ by means of the function λ defined by

$$\begin{aligned}\lambda(\tau) &= 0 & \text{if } \tau \in [\theta] \\ &= 1/e^x & \text{if } \tau = f_x = x^x \\ &= 1 & \text{if } \tau \in [1].\end{aligned}$$

From this it follows that T^{**} is isomorphic to T .

Theorem 4. *Let S be a standard thread with m the induced Lebesgue measure on S , and let S^* be the semigroup of equivalence classes of bounded, measurable, real-valued semicharacters.*

*Then S^{**} is isomorphic to S if and only if S has no non-idempotent nilpotent elements.*

Proof. As an immediate consequence of Clifford's theorems, we infer that a standard thread with no non-idempotent nilpotents is the ordinal sum $S = \sum_{e \in E} S'_e$ where $\{S'_e\}_{e \in E}$ are either unit threads or one element semigroups. We shall prove that the dual S^* of such an ordinal sum is the ordinal sum over the order dual, E^0 , of the duals $(S'_e)^*$; that is, $S^* = \sum_{e \in E^0} (S'_e)^*$. The theorem then follows immediately from the preceding corollary and the trivial fact that $(E^0)^0 = E$.

We begin by proving that any element of S^* either belongs to the equivalence class of the characteristic function $X_{[e_x, e_m]}$ for some $e_x \in E$ (where e_m denotes the maximal element of S); or is a natural extension of an element of $(S'_e)^*$ for some $S'_e \neq \{e\}$. This is done in the following lemmas.

Lemma 5. *If we fix a semicharacter ψ_0 in the dual of S and define $U\psi_0 = \{e : e \in E, \psi_0 \mid S'_e \not\equiv 0\}$, $L\psi_0 = \{e : e \in E, \psi_0 \mid S'_e \equiv 0\}$, then $\{L\psi_0, U\psi_0\}$ is a cut for E .*

Proof. i) Suppose $e < f$ with $f \in L\psi_0$. For any element $a \in S'_e$ and $b \in S'_f$, since $ab = ba = a$, we have $\psi_0(a) = 0$ which implies $e \in L\psi_0$.

ii) Similarly if $e \in U\psi_0$ with $f > e$ one can show $f \in U\psi_0$.

Remark. It also follows from (ii) that if $e, f \in U\psi_0$ with $f > e$, then $\psi_0 \mid S'_f \equiv 1$.

Lemma 6. *If $\{L\psi_0, U\psi_0\}$ is the cut defined in Lemma 5, then $\{L\psi_0, U\psi_0\}$ determines a point e_{x_0} of E .*

Proof. Case 1. Suppose $U\psi_0$ has no least element. Then, from the remark above, it follows that $\psi_0 \equiv 1$ on $U\psi_0$. If we let e_x denote the greatest element of $L\psi_0$ it follows that $S'_{e_x} = \{e_x\}$ and hence $\psi_0 = \chi(e_x, e_m)$.

Case 2. Suppose $L\psi_0$ has no greatest element whence $U\psi_0$ has a least element e_β . This least element must then be a one element semigroup and thus $\psi_0 = \chi[e_\beta, e_m]$.

Case 3. Suppose $L\psi_0$ and $U\psi_0$ have greatest and least elements e_α and e_β respectively. Then, since $\psi_0(x) = 0$ for $x \leq e_\alpha$ and $\psi_0(x) = 1$ for $x \geq e_\beta$ we conclude that ψ_0 is the natural extension of a semicharacter τ on S'_{e_α} . By natural extension we mean the semicharacter $\hat{\tau}$ on S obtained from τ in the dual of S'_{e_α} by

$$\begin{aligned}\hat{\tau}(x) &= 0 && \text{if } x \in S'_e \quad e < e_\alpha \\ &= (x) && \text{if } x \in S'_{e_\alpha} \\ &= 1 && \text{if } x \in S'_e \quad e > e_\alpha.\end{aligned}$$

Combining the three cases we see that any element $\psi \in S^*$ can be identified via the cut $\{L\psi, U\psi\}$ as belonging to the equivalence class of a characteristic function of the form $\chi_{[e_\alpha, e_m]}$, or as being the natural extension of a semicharacter τ on a thread associated with some initial point of E .

We now return to the proof of Theorem 4. Suppose that ψ_e and τ_f are two elements of S^* where $e, f \in E$ and $\psi_e \in (S'_e)^*$, $\tau_f \in (S'_f)^*$. Then we can define the pointwise product $\psi_e \circ \tau_f$ as follows

$$\begin{aligned}\psi_e \circ \tau_f &= \text{usual product} && \text{if } e = f \\ &= \psi_e \circ \tau_f = \tau_f \circ \psi_e = \tau_f && \text{if } e < f.\end{aligned}$$

However, since $S^* = \bigcup_{e \in E} (S'_e)^*$ where $\chi_{[e_\alpha, e_m]}$ is the only element of $(S'_e)^*$ if $S'_{e_\alpha} = \{e_\alpha\}$ we conclude that, as a semigroup, S^* is the ordinal sum over E^0 of the duals $(S'_e)^*$; that is,

$$S^* = \sum_{e \in E^0} (S'_e)^*$$

and thus

$$S^{**} = \sum_{e \in (E^0)^0} (S'_e)^{**}.$$

However, from Corollary 2 and the trivial relation $(E^0)^0 = E$ it follows that

$$S^{**} = \sum_{e \in E} (S'_e) = S.$$

The proof of the converse is much easier since the existence of a non-idempotent nilpotent implies the existence of a nil thread in the ordinal sum. Since the dual of a nil thread is a two point semigroup, it follows that the second dual is not isomorphic to the thread.

Bibliography

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