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EQUATION $Z' = A(t) - Z^2$ COEFFICIENT OF WHICH HAS A SMALL MODULUS

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In the paper [1] asymptotic properties of solutions of the equation

$$Z' = A(t) - Z^2 \tag{1}$$

were studied, $A(t)$ being a continuous complex-valued function defined on the interval $I = \langle t_0, \infty)$. Sufficient conditions were derived under which the trajectories of this equation behave like those of the equation

$$Z' = A - Z^2\,, \quad A = \text{const} \neq 0$$

near $t = \infty$.

In this note we will study the excluded case, that means the case when the function $A(t)$ has a small modulus. We will use the same method as in [1] in our investigations.

In what follows let R denote the set of all real numbers and K the set of all complex ones. If $Z = u + iv \in K$, we denote $\operatorname{Re} Z = u$, $\operatorname{Im} Z = v$, $\bar{Z} = u - iv$, $|Z| = \sqrt{(Z\bar{Z})}$. A curve $Z = Z(t)$ in the argand plane (u, v) is called the trajectory of the equation (1) on an interval i if and only if the function $Z(t)$ satisfies this equation on i.

Let a family of circles

$$\gamma = \frac{\Lambda\bar{Z} + \bar{\Lambda}Z}{Z\bar{Z}} \tag{2}$$

be given where $\Lambda \in K$, $\Lambda \neq 0$ and γ is a real parameter, $\gamma \in (-\infty, \infty)$. This equation can be written in the form $|\gamma Z - \Lambda| = |\Lambda|$ and represents a parabolic pencil of circles with the radical axis $\Lambda\bar{Z} + \bar{\Lambda}Z = 0$ corresponding to $\gamma = 0$. The circle K_γ corresponding to the value $\gamma \neq 0$ has the centre Λ/γ and the radius $r = |\Lambda|/\gamma$. For $\gamma \to \pm\infty$ the radius of K_γ converges to zero. The differential equation of the pencil (2) is of the form

$$\operatorname{Re} \Lambda\bar{Z}^2 Z' = 0$$

or

$$Z' = iv\bar{\Lambda}Z^2 \tag{3}$$

where $\nu \neq 0$ is a real constant. If $\operatorname{Re} \Lambda = 0$, $\nu \operatorname{Im} \Lambda = -1$, then the equation (3) becomes

$$Z' = -Z^2 . \tag{4}$$

These considerations imply the following

Lemma 1. *The trajectories of the equation* (4) *form a parabolic pencil of circles having the real axis for the radical one and cutting all curves* (2) *at the same angle* φ *for which*

$$\cos \varphi = \operatorname{Im} \Lambda / |\Lambda| , \quad \sin \varphi = \operatorname{Re} \Lambda / |\Lambda| .$$

(See Fig. 1)

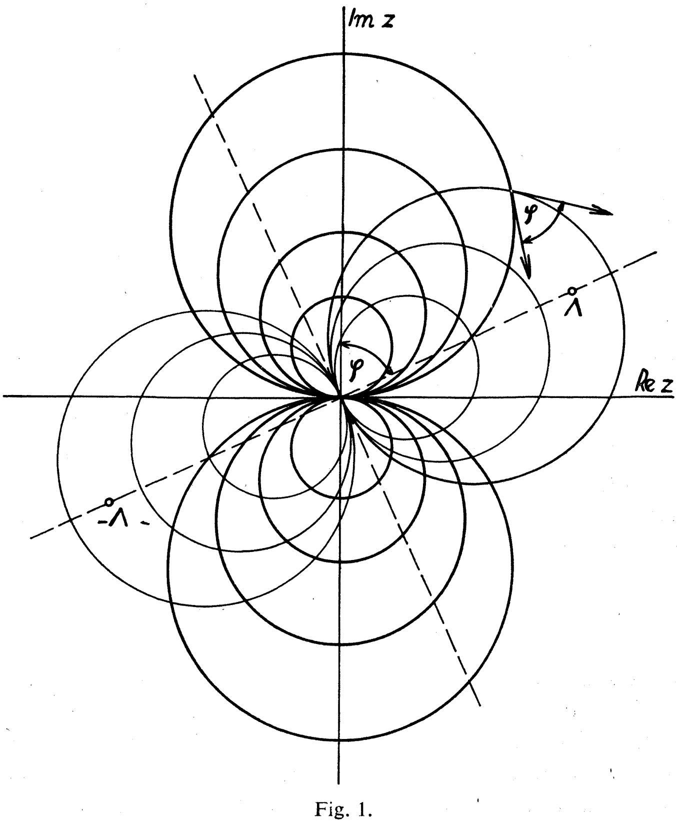

Fig. 1.

Lemma 2. *Let* $A(t)$ *be continuous on* I *and let* $Z = Z(t)$ *be a trajectory of* (1). *Suppose that there is a* $\Lambda \in K$, $\operatorname{Re} \Lambda \geqq 0$ *such that*

$$\Lambda \bar{A}(t) + \bar{\Lambda} A(t) > 0 \tag{5}$$

for all $t \in I$ and denote $M(\Lambda) = \{Z \in K : \Lambda\bar{Z} + \bar{\Lambda}Z > 0\}$. If there is a time $t = T$ such that $Z(T) \in M(\Lambda)$, then $Z(t) \in M(\Lambda)$ for all $t \in I$.

Proof. It is sufficient to prove that at every time T for which $H(T) = \Lambda\bar{Z}(T) + \bar{\Lambda}Z(T) = 0$ it holds $H'(T) > 0$. The relation $H(T) = 0$ implies $\operatorname{Re}[\Lambda\bar{Z}(T)] = 0$ and $\Lambda^2\bar{Z}^2(T) = \bar{\Lambda}^2Z^2(T) = -(\operatorname{Im}[\Lambda\bar{Z}(T)])^2$, so that

$$H'(T) = \Lambda\bar{Z}'(T) + \bar{\Lambda}Z'(T) = \Lambda[\bar{A}(T) - \bar{Z}^2(T)] + \bar{\Lambda}[A(T) - Z^2(T)] =$$

$$= \Lambda\bar{A}(T) + \bar{\Lambda}A(T) - \frac{\bar{\Lambda}}{\Lambda\bar{\Lambda}}\Lambda^2\bar{Z}^2(T) - \frac{\Lambda}{\Lambda\bar{\Lambda}}\bar{\Lambda}^2Z^2(T) =$$

$$= \Lambda\bar{A}(T) + \bar{\Lambda}A(T) + \frac{2}{\Lambda\bar{\Lambda}}(\operatorname{Im}[\Lambda\bar{Z}(T)])^2 \operatorname{Re}\Lambda > 0\,.$$

The proof is complete.

Theorem 1. *Let $A(t)$ be a continuous complex-valued function of the real variable $t \in I$ with bounded modulus. Let*

$$\sup_{t\in I} |A(t)| = \delta\,. \tag{6}$$

Suppose that there exists a $\Lambda \in K$, $\operatorname{Re}\Lambda > 0$ satisfying (5). Let $\varrho \in R$,

$$\varrho > \sqrt{\frac{|\Lambda|\,\delta}{\operatorname{Re}\Lambda}}\,. \tag{7}$$

If $Z = Z(t)$ is a trajectory of (1) satisfying at a time $t_1 \geqq t_0$ the condition $Z(t_1) \in M(\Lambda)$, then $Z(t) \in M(\Lambda)$ for all $t \geqq t_1$ and

$$\liminf_{t\to\infty} |Z(t)| < \varrho\,.$$

Proof. Every trajectory $Z = Z(t)$ of (1) satisfying at $t = t_1$ the condition $Z(t_1) \in M(\Lambda)$, remains in the halfplane

$$\Lambda\bar{Z} + \bar{\Lambda}Z > 0\,.$$

This is the consequence of Lemma 1. The circle of the pencil (2) passing through the point $Z(t)$ corresponds to the value $\gamma(t)$

$$\gamma(t) = \frac{\Lambda\bar{Z}(t) + \bar{\Lambda}Z(t)}{Z(t)\,\bar{Z}(t)}\,. \tag{8}$$

Differentiation yields

$$\gamma'(t) = -2\operatorname{Re}\frac{\Lambda Z^2(t)\,[A(t) - Z^2(t)]}{Z^2(t)\,\bar{Z}^2(t)}\,,$$

so that

$$\tfrac{1}{2}\gamma'(t) = \operatorname{Re} \Lambda - \operatorname{Re} \frac{\Lambda A(t)}{Z^2(t)} . \tag{9}$$

In contradiction to the assertion of Lemma 1 suppose $\liminf_{t\to\infty} |Z(t)| \geqq \varrho$. Choose an $\varepsilon > 0$ in such a way that $\varrho - \varepsilon > (|\Lambda| \delta / \operatorname{Re} \Lambda)$. Then there is a $T \in I$ such that

$$|Z(t)| > \varrho - \varepsilon \quad \text{for all} \quad t \geqq T \tag{10}$$

and from (6), and (9) we have the following inequality:

$$\tfrac{1}{2}\gamma'(t) > \operatorname{Re} \Lambda - \frac{|\Lambda| \delta}{(\varrho - \varepsilon)^2} > 0 .$$

Hence we have

$$\gamma(t) > \gamma(T) + 2\left[\operatorname{Re} \Lambda - \frac{|\Lambda| \delta}{(\varrho - \varepsilon)^2}\right] t \to \infty \quad \text{for} \quad t \to \infty$$

and this means that the radius of the circle $K_{\gamma(t)}$ of the pencil (2) converges to zero. From this fact the existence of a time $t_1 \geqq T$ follows such that $|Z(t_1)| = \varrho - \varepsilon$, which contradicts to (10).

The proof is complete.

Consequence. *Let $A(t)$ be a continuous function on I, $\lim_{t\to\infty} A(t) = 0$. Suppose that there exists a $\Lambda \in K$, $\operatorname{Re} \Lambda > 0$ such that (5) holds. Let $Z = Z(t)$ be trajectory of (1) satisfying the condition $Z(t_1) \in M(\Lambda)$ at a time $t_1 \geqq t_0$. Then $Z(t) \in M(\Lambda)$ for all $t > t_1$ and*

$$\liminf_{t\to\infty} |Z(t)| = 0 .$$

Theorem 2. *Let $A(t)$ be continuous on I. Let linearly independent $\Lambda_1, \Lambda_2 \in K$ exist in such a way that*

$$\operatorname{Re} \Lambda_i > 0 , \quad \Lambda_i A(t) + \bar{\Lambda}_i A(t) > 0 , \quad i = 1, 2 . \tag{11}$$

Denote $\Lambda = \frac{1}{2}(\Lambda_1 + \Lambda_2)$, $M = M(\Lambda_1) \cap M(\Lambda_2)$. If $Z = Z(t)$ is any trajectory of (1) satisfying at $t_1 \geqq t_0$ the condition $Z(t_1) \in M$, then $Z(t) \in M$ for all $t \geqq t_1$.

Let the modulus of $A(t)$ be bounded on I.

Let

$$\delta = \sup_{t \geqq t_1} |A(t)| , \quad L = \max_{i=1,2} \frac{|\Lambda_i|^2}{(\Lambda_i \bar{\Lambda} + \bar{\Lambda}_i \Lambda)^2} ,$$

$$\gamma_0 \in R , \quad 0 < \gamma_0 < \sqrt{\left(\frac{\operatorname{Re} \Lambda}{L |\Lambda| \delta}\right)} . \tag{12}$$

Then there exists a time $t_2 \geqq t_1$ such that the point $Z(t)$ remains in the interior of the circle K_{γ_0} of the pencil (2) for all $t > t_2$.

Proof. Let $Z = Z(t)$ be any trajectory of (1) satisfying at $t = t_1$ the condition $Z(t_1) \in M$. Then $Z(t) \in M$ for all $t > t_1$ in view of Lemma 2. The circle of the pencil (2) passing through the point $Z(t)$ corresponds to the value $\gamma(t)$ given by means of (8) and its derivative is (9).

Let

$$R(t) = \left| \operatorname{Re} \frac{\Lambda A(t)}{Z^2(t)} \right| .$$

Then we have

$$R(t) \leqq \gamma^2(t)\, |\Lambda\, A(t)| \left| \frac{Z(t)\Lambda}{\overline{Z(t)} + \bar{\Lambda}\, Z(t)} \right|^2 .$$

The function

$$f(Z) = \left| \frac{Z}{\Lambda\bar{Z} + \bar{\Lambda} Z} \right|^2$$

assumes the constant value

$$f(Z)|_{H\bar{Z}+\bar{H}Z=0} = \left| \frac{H\bar{H}Z}{H\bar{H}\Lambda\bar{Z} + H\bar{H}\bar{\Lambda}Z} \right|^2 = \left| \frac{H}{H\bar{\Lambda} - \bar{H}\Lambda} \right|^2 = F(H)$$

on the line $H\bar{Z} + \bar{H}Z = 0$, $H \in K$, $H \neq 0$.

Consider the values of the function $F(H)$ on the circle $|H| = |\Lambda|$. Here, the function $F(H)$ is defined for all $H \neq \pm\Lambda$, is positive, reaches its minimum $\frac{1}{4}|\Lambda|^{-2}$ for $H = \pm i\Lambda$ and $F(H) \to \infty$ for $H \to \pm\Lambda$. Then, in the domain M, the function $f(Z)$ reaches its greatest value on one of the lines $\Lambda_i\bar{Z} + \bar{\Lambda}_i Z = 0$, $i = 1, 2$. Thus, at every point $Z(t) \in M$, the following inequality holds:

$$R(t) \leqq L|\Lambda|\, |A(t)|\, \gamma^2(t) ,$$

so that

$$\operatorname{Re}\Lambda - L|\Lambda\, A(t)|\, \gamma^2(t) \leqq \tfrac{1}{2}\gamma'(t) \leqq \operatorname{Re}\Lambda + L|\Lambda\, A(t)|\, \gamma^2(t) . \tag{13}$$

Let $\Gamma_0 = 2[\operatorname{Re}\Lambda - L|\Lambda|\, \delta\gamma_0^2]$. According to (12) we have

$$\gamma'(t) \geqq \Gamma_0 > 0 \tag{14}$$

for $\gamma < \gamma_0$.

If $\gamma(t_1) \geqq \gamma_0$, then $\gamma(t) > \gamma_0$ for all $t > t_1$; this is a consequence of the fact that $\gamma(t) = \gamma_0$ implies $\gamma'(t) > 0$ with respect to (14). If $\gamma(t_1) < \gamma_0$, we proceed as follows:

assuming $\gamma(t) < \gamma_0$ for all $t > t_0$ and integrating (14) from t_1 to t we get

$$\gamma(t) > m + \Gamma_0 t$$

where $m = \gamma(t_1) - \Gamma_0 t_1$, so that $\gamma(t) \to \infty$ for $t \to \infty$ which contradicts to the fact that $\gamma(t) < \gamma_0$. Consequently, there is a $t_2 > t_1$ such that $\gamma(t_2) = \gamma_0$ and $\gamma(t) > \gamma_0$ for all $t - t_2$.

Consequence. *Let $A(t)$ be continuous on I, $\lim_{t\to\infty} A(t) = 0$. Suppose that linearly independent constants $\Lambda_1, \Lambda_2 \in K$ exist satisfying (11). Let Λ and M have the same meaning as in the preceding Theorem.*

Then every trajectory $Z = Z(t)$ of the equation (1) satisfying at a time $t_1 \geqq t_0$ the condition $Z(t_1) \in M$ converges to the origin in such a way that it remains in M for all $t > t_1$.

Note. If for example $A(t) = a(t) + ib(t)$, $\Lambda = \lambda + i\mu$ and $a(t) \geqq 0$, $b(t) \geqq 0$, $a^2(t) + b^2(t) > 0$, then the inequality

$$\Lambda\, \bar{A}(t) + \bar{\Lambda}\, A(t) = \lambda\, a(t) + \mu\, b(t) > 0 \tag{15}$$

holds for all $\lambda > 0$, $\mu > 0$. That means: every trajectory $Z = Z(t)$ of the equation (1) starting at a point Z of the domain $M = \{Z \in K : \operatorname{Re} Z > 0 \text{ or } \operatorname{Im} Z > 0\}$ remains for $t \to \infty$ in M.

Theorem 3. *Assume $A(t)$ to be continuous on I and*

$$\int_{t_0}^{\infty} |A(t)|\, \mathrm{d}t < \infty\,. \tag{16}$$

Let linearly independent $\Lambda_1, \Lambda_2 \in K$ exist satisfying (11). Let Λ and M preserve their meaning from Theorem 1. Then every trajectory of (1) satisfying at a time $t_1 \geqq t_0$ the condition $Z(t_1) \in M$ remains in M for all $t > t_1$ and

$$\lim_{t\to\infty} Z(t) = 0\,.$$

Proof. Let $Z = Z(t)$ be a trajectory of (1) starting at a point of M. In view of Lemma 2, $Z(t) \in M$ for all $t > t_1$. The circle of the pencil (2) passing through the point $Z(t)$ corresponds to the value $\gamma(t)$ for which the inequalities (13) hold.

Therefore

$$-L|\Lambda\, A(t)| \leqq \frac{1}{2}\frac{\gamma'(t)}{\gamma^2(t)} - \frac{\operatorname{Re} \Lambda}{\gamma^2(t)} \leqq L|\Lambda\, A(t)|\,. \tag{17}$$

Integrating these inequalities over $\langle t_1, t\rangle$ we see, according to (16), that the assumption $\int_{t_1}^{\infty} (\mathrm{d}t/\gamma^2(t)) = \infty$ would imply

$$\lim_{t\to\infty} \int_{t_1}^{t} \frac{\gamma'(s)}{\gamma^2(s)}\,\mathrm{d}s = \lim_{t\to\infty} [\gamma^{-1}(t_1) - \gamma^{-1}(t)] = \infty$$

which contradicts to the fact that $\gamma(t) > 0$. Therefore

$$\int_{t_1}^{\infty} \frac{\mathrm{d}t}{\gamma^2(t)} < \infty\,. \tag{18}$$

Moreover, from (17) we obtain

$$\frac{1}{2}\left|\frac{\gamma'(t)}{\gamma^2(t)}\right| \leqq \frac{\operatorname{Re} \Lambda}{\gamma^2(t)} + L|\Lambda\, A(t)|\,.$$

Hence and from (18) we have

$$\int_{t_1}^{\infty} \left|\frac{\gamma'(t)}{\gamma^2(t)}\right| \mathrm{d}t < \infty\,.$$

This implies the existence of a finite limit $\lim_{t\to\infty} \gamma^{-1}(t)$ and with respect to (18) it holds $\gamma(t) \to \infty$. But this means $\lim_{t\to\infty} Z(t) = 0$ and the proof is complete.

References

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