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NOTE ON SEPARATION OF CONVEX SETS

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A statement is proved concerning separation of two convex sets by two disjoint balls.

We work in real Banach spaces. $K_r(x)$ denotes the closed ball centered at x with radius r. By a ball we shall always mean a closed one. The set of all real numbers is denoted by R, X^* means the dual space of the Banach space X with the usual supremum-norm on $K_1(0) \subset X$. $S_r(x)$ denotes the norm boundary of $K_r(x) \subset X$. $K_r^*(0) = \{f \in X^*; \|f\| \leqq r\}$, $S_r^* = \{f \in X^*; \|f\| = r\}$.

Following R. R. PHELPS ([8]), we shall call $f \in S_1^*$ the strongly exposed point of K_1^* if f attains its norm at the point $x \in S_1$ which is a point of strong (Fréchet) differentiability of $\|x\|$ of X. The set of all strongly exposed points of K_1^* will be denoted by str K_1^*. For $A \subset X$, $\delta(A)$ denotes the norm boundary of A in X. For $f \in X^*$, $A \subset X$, $f(A) \leqq c$ means $f(y) \leqq c$ for $y \in A$.

Definition 1. Let K be a convex subset of X, $z \in \delta(K)$. $f \in X^*$ is said to be *the supporting functional* of K at z if either $f(k) \geqq f(z)$ for all $k \in K$ or $f(k) \leqq f(z)$ for all $k \in K$.

Lemma. *Let* $0 \neq f \in X^*$ *be a supporting functional of* $K_r(0)$ *at* $x \in \delta(K_r(0))$. *Take any* $p > 0$ *and* $z \in Rx$. *Then*:

1) f *is a supporting functional of* $K_p(z)$ *at both points* $\delta(K_p(z)) \cap Rx$.

2) $0 \notin K_p(z)$ *implies either* $f(k) > 0$ *for all* $k \in K_p(z)$ *or* $f(k) < 0$ *for all* $k \in K_p(z)$.

Proof. 1) First assume $z = 0$, $p > 0$. Then it is easy to verify that f is a supporting functional of $K_p(0)$ at $\pm(p/r)\,x$. In fact, take for instance the point $(p/r)\,x$. Assume, without a loss of generality, $f(y) \geqq f(x)$ for $y \in K_r(0)$. Then if $y \in K_p(0)$, it is $(r/p)\,y \in K_r(0)$ and therefore $f((r/p)\,y) \geqq f(x)$, i.e. $f(y) \geqq f((p/r)\,x)$.

Now, take $p > 0$, $\alpha \in R$, $z = \alpha x$. $\delta(K_p(z)) \cap Rx = (\alpha \pm p/r)\,x$. Take for instance $(\alpha + p/r)\,x$. Let $Ay = y - \alpha x$ for $y \in X$. Then $AK_p(\alpha x) = K_p(0)$. Since f is a supporting functional of $K_p(0)$ at $(p/r)\,x$ we have either $f(y - \alpha x) \geqq f((p/r)\,x)$ for each $y \in K_p(\alpha x)$ or $f(y - \alpha x) \leqq f((p/r)\,x)$ for each $y \in K_p(\alpha x)$. Hence either $f(y) \geqq f((p/r)\,x + \alpha x)$ for $y \in K_p(\alpha x)$ or $f(y) \leqq f((\alpha + p/r)\,x)$ for each $y \in K_p(\alpha x)$ which means f is a supporting functional of $K_p(\alpha x)$ at $(\alpha + p/r)\,x$.

2) If $f(x) = 0$ then for all $y \in K_r(0)$ we should have either $f(y) \leqq 0$ for $y \in K_r(0)$ or $f(y) \geqq 0$ for all $y \in K_r(0)$. Both cases are obviously impossible, therefore $Rx \cap \cap f^{-1}(0) = \{0\}$. Thus the fact $0 \notin K_p(z)$ is equivalent to the fact that $K_p(z) \cap Rx \cap \cap f^{-1}(0) = \emptyset$. Denote $\delta(K_p(z)) \cap Rx = \{v, w\}$. Suppose $f(v) > 0$. Then $f(w) > 0$, otherwise there exists $\alpha_0 x \in K_p(z)$, $f(\alpha_0 x) = 0$. Then $\alpha_0 x \in f^{-1}(0) \cap Rx \cap K_p(z)$ which is a contradiction with our assumption $0 \notin K_p(z)$. We have also $f(w) \neq f(v)$, since f is one-to-one on Rx. Assume without any loss of generality $f(w) > f(v)$. Since f is a supporting functional of $K_p(z)$ at v, we have $f(y) \geqq f(v) > 0$ for all $y \in K_p(z)$. Similarly for $f(v) < 0$. The following statement was motivated by the results of S. Mazur ([7]) and R. R. Phelps ([8]):

Proposition 1. *Assume X is a Banach space such that* str $K_1^* \neq \emptyset$. *Let K be a convex closed bounded subset of X, $f \in$* str K_1^* *so that* $\inf f(K) > 0$. *Then there exists a ball $B \subset X$, $B \supset K$ so that $f(B) > 0$.*

Proof. Let $x \in S_1$ be such that $f(x) = 1$, $\|x\|$ of X is strongly differentiable at x. Let us choose $\varepsilon > 0$ so that $\inf f(K) > 2\varepsilon > 0$. Take $z = \varepsilon x$. Now, following S. Mazur ([7]), take a system $\mathscr{K}$ of balls: $K_{(r-1)\varepsilon}(rz)$ for $r > 1$.

Then it is possible to prove ([7]) that, while $0 \notin K_{(r-1)\varepsilon}(rz)$ for all $r > 1$, there exists $r_0 > 1$ so that $K \subset K_{(r_0-1)\varepsilon}(r_0 z)$. Repeat, for completeness, this proof:

The first statement is obvious.

For the second one, suppose there exist sequences $\{r_n\}$ and $\{x_n\}$ so that $r_n > 1$, $r_n \to \infty$, $\|x_n - r_n z\| > (r_n - 1)\varepsilon$, $x_n \in K$ for each n. Denote $y_n = -x_n/\varepsilon r_n$. Then $y_n \to 0$. We have $\|x + y_n\| - \|x\| = D\|\,.\,\|(x, y_n) + \omega(y_n)$, $\omega(y_n)/\|y_n\| \to 0$ (where $D\|\,.\,\|(x, h)$ denotes the differential of $\|\,.\,\|$ and ω the remainder), since $\|\,.\,\|$ is strongly differentiable at x. We have

$$\left\|x - \frac{x_n}{\varepsilon r_n}\right\| - 1 = f(y_n) + \omega(y_n)$$

so that $\varepsilon r_n\, \omega(y_n) = \|x_n - r_n z\| - \varepsilon r_n + f(x_n) > (r_n - 1)\varepsilon - \varepsilon r_n + 2\varepsilon = \varepsilon$. Hence

$$\frac{\omega(y_n)}{\|y_n\|} = \frac{\varepsilon r_n\, \omega(y_n)}{\|x_n\|} > \frac{\varepsilon}{\|x_n\|} \nrightarrow 0$$

since $\{x_n\}$ is bounded. Therefore we have a contradiction with Fréchet differentiability of $\|\,.\,\|$ at x. Thus there exists $r_0 > 1$ such that $K \subset K_{(r_0-1)\varepsilon}(r_0 z)$. Now, we may apply our lemma on $\mathscr{K}, f$ and see that since $0 \notin K_{(r_0-1)\varepsilon}(r_0 z)$ and $f(r_0 z) > 0$ we have $f(k) > 0$ for all $k \in K_{(r_0-1)\varepsilon}(r_0 z)$.

Corollary. *Suppose a Banach space X has the property that* str K_1^* *is a norm dense in S_1^*. K_1, K_2 be closed convex bounded subsets of X, one of them being weakly compact. Then there exist balls B_1, B_2 so that $B_i \supset K_i$, $i = 1, 2$, $B_1 \cap B_2 = \emptyset$.*

Proof. By the well known Separation Theorem ([3]) there exist $f \in S_1^*$, $\varepsilon > 0$, $c \in R$ so that $f(K_1) \leqq c - \varepsilon < c < f(K_2)$.

Take $c_1 = c - \frac{1}{2}\varepsilon$. Then $\sup f(K_1) < c_1 - \frac{1}{4}\varepsilon < c_1 + \frac{1}{4}\varepsilon < \inf f(K_2)$. We may choose $\tilde{f} \in \operatorname{str} K_1^*$ so that $\sup \tilde{f}(K_1) < c_1 < \inf \tilde{f}(K_2)$. First, consider K_2. Let $z \in X$ be such that $\tilde{f}(z) = c_1$. Then consider a translation $Ay = y - z$ for $y \in X$. Denote $\tilde{K}_2 = AK_2$. $\tilde{K}_2$ is a closed convex bounded set, $\inf \tilde{f}(\tilde{K}_2) > 0$. By our proposition there exists a ball $\tilde{B} \supset \tilde{K}_2$ so that $\tilde{f}(\tilde{B}) > 0$. $A^{-1}\tilde{B} = B$ is then a ball so that $B \supset K_2$, $\tilde{f}(B) > c_1$. Analogously, dealing with $-\tilde{f}(\in \operatorname{str} K_1^*)$ we may obtain a ball $B_1 \supset K_1$, $\tilde{f}(B_1) < c_1$. Therefore $B_1 \cap B_2 = \emptyset$.

In this connection, perhaps, the following fact is worth mentioning, too:

It is almost obvious that whenever $\|.\|$ is Gâteaux differentiable at $x_0 \in S_1 \subset X$ then the limit

$$\lim_{t \to 0} \frac{\|x_0 + th\| - \|x_0\|}{t} = D\|.\|(x_0, h)$$

is uniform on $h \in K$ where K is an arbitrary norm compact subset of X. To prove it (as for example N. A. Ivanov) [3a])) suppose this is not true for some compact $K \subset X$. Then there exist $t_n \to 0$, $h_n \in K$ such that whenever we write $\|x_0 + th\| - \|x_0\| = D\|.\|(x_0, th) + \omega(x_0, th)$, then

$$\left|\frac{\omega(x_0, t_n h_n)}{t_n}\right| \geqq \varepsilon > 0\,.$$

Without any loss of generality suppose $h_n \to h \in K$. Then

$$\begin{aligned}\left|\frac{\omega(x_0, t_n h)}{t_n}\right| &= \left|\frac{\omega(x_0, t_n h_n)}{t_n} + \frac{\|x_0 + t_n h\| - \|x_0 + t_n h_n\|}{t_n} + \right.\\ &\quad \left. + D\|.\|(x_0, h_n) - D\|.\|(x_0, h)\right| \geqq \\ &\geqq \left|\frac{\omega(x_0, t_n h_n)}{t_n}\right| - \left(\left|\frac{\|x_0 + t_n h\| - \|x_0 + t_n h_n\|}{t_n}\right| + \right.\\ &\quad \left. + \left|D\|.\|(x_0, h_n) - D\|.\|(x_0, h)\right|\right) \geqq \\ &\geqq \left|\frac{\omega(x_0, t_n h_n)}{t_n}\right| - (\|h_n - h\| + \|h_n - h\|) \geqq \\ &\geqq \frac{\varepsilon}{2}\end{aligned}$$

for $n \geqq n_0$ — a contradiction with Gâteaux differentiability of $\|.\|$ at x_0.

Definition 2. Call $f \in S_1^*$ the X-exposed point of K_1^* if there exists $x \in S_1$ such that $f(x) = 1$ and $\|x\|$ is Gâteaux differentiable at x. The set of all X-exposed points of K_1^* denote by $\exp_X K_1^*$.

Analogously to Proposition 1 we may derive:

Proposition 2. *Assume X is a Banach space such that* $\exp_X K_1^* \neq \emptyset$. *Let K be a compact convex subset of X, $f \in \exp_X K_1^*$ so that* $\inf f(K) > 0$. *Then there exists a ball $B \subset X$, $B \supset K$ such that $f(B) > 0$.*

Proof. Follow the proof of Proposition 1; put further $t_n = 1/\varepsilon r_n$, $h_n = -x_n$. Then we have $t_n \to 0$,

$$\left|\frac{\omega(t_n h_n)}{t_n}\right| \geqq \frac{\varepsilon t_n}{t_n} = \varepsilon$$

– a contradiction. Therefore, we again have

Corollary. *Suppose a Banach space X has the property that* $\exp_X K_1^*$ *is a norm dense on S_1^*, K_1, K_2 be two disjoint compact convex sets in X. Then there exist two balls $B_i \supset K_i$, $i = 1, 2$, $B_1 \cap B_2 = \emptyset$.*

As for the assumptions of our propositions we would like to remark the following:

First, the Bishop-Phelps Theorem ([2]) says that for every Banach space X the set C of all continuous linear functionals on X which attain their norms on $S_1 \subset X$ is norm-dense in X^*. Therefore if we suppose $\|x\|$ of X is Fréchet (Gâteaux) differentiable at every $x \in S_1$ we have immediately $\operatorname{str} K_1^* = C \cap S_1^*$ $(\exp_X K_1^* = C \cap S_1^*)$. Thus our assumptions as for the density of strongly exposed (X-exposed) points of K_1^* are satisfied if $\|x\|$ of X is Fréchet (Gâteaux) differentiable on $S_1 \subset X$.

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References

[1] *E. Asplund:* Frèchet differentiability of convex functions, Acta Mathem. *121* (1–2), (1968), 31–48.

[2] *E. Bishop, R. R. Phelps:* A proof that every Banach space is subreflexive, Bull. Am. Math. Soc. *67*, 79–98.

[3] *N. Dunford, J. T. Schwartz:* Linear operators. Part I, Interscience, New York 1966.

[3a] *Н. А. Иванов:* О дифференциалах Гато и Фреше, Успехи Матем. наук *10* (1955), 161–166.

[4] *M. I. Kadec:* On the spaces isomorphic to locally uniformly rotund spaces (Russian), Izvestija vysš. uč. zav. Matem. 1, (1959), 6, 51–57 and 1, (1961), 6, 186–187.

[5] *J. Lindenstrauss:* On operators which attain their norm, Israel Journal Math. *1* (1963), 139–148.

[6] *A. R. Lovaglia:* Locally uniformly convex Banach spaces, Trans. Am. Math. Soc. *78* (1958) 225–238.

[7] *S. Mazur:* Über schwache Konvergenz in den Räumen L^p, Studia Math. *4* (1933) 128–133.

[8] *R. R. Phelps:* A representation theorem for bounded convex sets, Proc. Am. Math. Soc. *11* (1960), 976–983.

[9] *V. L. Šmuljan:* Sur la structure de la sphere unitaire dans l'espace de Banach, Matem. Sbornik *9* (1941) 545–572.

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