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REMARKS ON A THEOREM OF P. K. SUETIN

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1. Let

$$S_n(x) = \sum_{k=0}^{n} a_k \tilde{P}_k(x)$$

denote the $nth$ partial sum of the Fourier Legendre series of a function $f(x)$. It is well-known that $S_n(x)$ converges uniformly to $f(x)$ in $[-1, 1]$ if $f(x)$ has a continuous second derivative on $[-1, 1]$. Recently Suetin [4] has shown that $S_n(x)$ converges uniformly to $f(x)$ if $f(x)$ belongs to a Lipschitz class of order greater than $1/2$ in $[-1, 1]$.

More precisely he has established the following result.

**Theorem 1.** (P. K. Suetin [4]). If $f(x)$ has $p$ continuous derivatives on $[-1, 1]$ and \( f^{(p)}(x) \in \text{Lip} \, \alpha \), then

$$|f(x) - S_n(x)| \leq \frac{c_1 \log n}{n^{p+\alpha - 1/2}}, \quad x \in [-1, 1],$$

for $p + \alpha \geq 1/2$.

In establishing this remarkable theorem he has employed the following well known theorem of A. F. Timan [6] which is a stronger form of Jackson’s theorem.

**Theorem 2.** If $f(x)$ has $p$ continuous derivatives on $[-1, 1]$ and \( f^{(p)}(x) \in \text{Lip} \, \alpha \), then there is a sequence of polynomials \( \{q_n(x)\} \) for which

$$|f(x) - q_n(x)| \leq \frac{c_2}{n^{p+\alpha}} \left( \sqrt{1 - x^2} + \frac{1}{n} \right)^{p+\alpha}, \quad x \in [-1, 1].$$

Very recently Saxena [3] has proved the following theorem for $S'_n(x)$, the first derivative of $S_n(x)$ with respect to $x$.  

349
Theorem 3 (R. B. Saxena [3]). If \( f(x) \) has \( p \) continuous derivatives on \([-1, 1]\) and \( f^{(p)}(x) \in \text{Lip } \alpha \), then together with (1.2) the following inequalities hold:

\[
(1.3) \quad (1 - x^2)^{3/4} |f'(x) - S_n'(x)| \leq \frac{c_3 \log n}{n^{p+\alpha-1}}, \quad (0 < \alpha < 1, \ p \geq 1),
\]

\[
(1.4) \quad (1 - x^2)^{1/2} |f'(x) - S_n'(x)| \leq \frac{c_4 \log n}{n^{p+\alpha-3/2}}, \quad (\frac{1}{2} < \alpha < 1, \ p \geq 1)
\]

and

\[
(1.5) \quad |f'(x) - S_n'(x)| \leq \frac{c_5 \log n}{n^{p+\alpha-5/2}}, \quad (\frac{1}{2} < \alpha < 1, \ p \geq 2)
\]

uniformly in \([-1, 1]\).

In connection with theorem 1 we shall prove the following theorem which generalizes theorem 3.

Theorem 4. If \( f(x) \) has \( p \) continuous derivatives on \([-1, 1]\) and \( f^{(p)}(x) \in \text{Lip } \alpha \), then together with (1.3) and (1.4) the following inequalities hold:

\[
(1.6) \quad (1 - x^2)^{1/4} |f(x) - S_n(x)| \leq \frac{c_6 \log n}{n^{p+\alpha}}, \quad (p + \alpha \geq \frac{1}{2})
\]

and

\[
(1.7) \quad |f^{(p)}(x) - S_n^{(p)}(x)| \leq \frac{c_7 \log n}{n^{p+\alpha-2r-1/2}}, \quad (p \geq 2r, \ \frac{1}{2} < \alpha < 1)
\]

uniformly in \([-1, 1]\).

2. To prove the theorem we shall need the following well-known results on Legendre polynomials. The orthonormalized Legendre polynomial \( \bar{P}_n(x) \) is given by [1]

\[
(2.1) \quad \bar{P}_n(x) = \sqrt{\frac{n + 1}{2}} P_n(x),
\]

where \( P_n(x) \) denotes the \( n \)th Legendre polynomial with the normalization \( P_n(1) = 1 \). From [1], [2] and [5] we have for \(-1 \leq x \leq 1\),

\[
(2.2) \quad |\bar{P}_n(x)| \leq c_7 \sqrt{n}
\]

and the inequality

\[
(2.3) \quad (1 - x^2)^{1/4} |\bar{P}_n(x)| \leq c_8.
\]
For the derivatives of $\bar{P}_n(x)$ we have the following inequalities which hold for $-1 \leq x \leq 1$,
\begin{align}
(1 - x^2)^{1/2} \left| \bar{P}_n^{(r)}(x) \right| & \leq c_0 n^{3/2}, \\
(1 - x^2)^{3/4} \left| \bar{P}_n^{(r)}(x) \right| & \leq c_{10} n
\end{align}
and the Markov's inequality
\begin{equation}
\left| \bar{P}_n^{(r)}(x) \right| \leq c_{11} n^{2r+1/2}, \quad r = 0, 1, 2, \ldots
\end{equation}

3. In order to prove Theorem 4 we require the following lemmas.

**Lemma 3.1.** For $-1 \leq x \leq 1$, we have
\begin{equation}
(1 - x^2)^{1/4} \int_{-1}^{1} \left| \sum_{k=0}^{n} \bar{P}_k(t) \bar{P}_k(x) \right| \, dt \leq c_{11} n^{1/2}
\end{equation}
and
\begin{equation}
\int_{-1}^{1} \left| \sum_{k=r}^{n} \bar{P}_k(t) \bar{P}_k^{(r)}(x) \right| \, dt \leq c_{12} n^{2r+1}.
\end{equation}

**Proof.** We give here the proof for (3.2) only. The proof for (3.1) can be given on the same lines. Making use of (2.6) we have
\[
\int_{-1}^{1} \left[ \sum_{k=r}^{n} \bar{P}_k(t) \bar{P}_k^{(r)}(x) \right]^2 \, dt = \sum_{k=r}^{n} \left| \bar{P}_k^{(r)}(x) \right|^2 \leq c_{13} \sum_{k=r}^{n} k^{4r+1} \leq c_{14} n^{4r+2},
\]
from which (3.2) follows.

**Lemma 3.2.** We have for $-1 \leq x \leq 1$ and $\alpha \geq 1/2$,
\begin{equation}
(1 - x^2)^{1/4} \int_{-1}^{1} \left( \sqrt{1 - t^2} \right)^{p+\alpha} \left| \sum_{k=0}^{n} \bar{P}_k(t) \bar{P}_k(x) \right| \, dt \leq c_{15} \log n
\end{equation}
and
\begin{equation}
\int_{-1}^{1} \left( \sqrt{1 - t^2} \right)^{p+\alpha} \left| \sum_{k=r}^{n} \bar{P}_k(t) \bar{P}_k^{(r)}(x) \right| \, dt \leq c_{16} n^{2r+1/2} \log n.
\end{equation}

**Proof.** We shall prove (3.4) only and (3.3) can be proved in the same manner. Let us denote by $A_n(x)$ the part of $[-1, 1]$ on which $|x - t| \leq 1/n$ and by $\delta(x)$ the rest of the interval. Making use of (2.3) and (2.6), we obtain
\begin{equation}
\int_{A_n(x)} (1 - t^2)^{(p+\alpha)/2} \left| \sum_{k=r}^{n} \bar{P}_k(t) \bar{P}_k^{(r)}(x) \right| \, dt \leq
\end{equation}
\[
\leq \int_{A_n(x)} \sum_{k=r}^{n} (1 - t^2)^{(p+\alpha)/2} \left| \bar{P}_k(t) \right| \left| \bar{P}_k^{(r)}(x) \right| \, dt \leq K_r \frac{1}{n} \sum_{k=0}^{n} k^{2r+1/2} \leq K_r n^{2r+1/2}.
\]

351
To estimate the integral over \( \delta_n(x) \) we make use of the Christoffel formula [5].

\[
(3.6) \quad \sum_{k=0}^{n} \bar{P}_k(t) \bar{P}_k(x) = \theta_n \frac{\bar{P}_{n+1}(t) - \bar{P}_n(t) \bar{P}_{n+1}(t)}{x - t}, \quad 0 < \theta_n \leq 1.
\]

On differentiating \( r \) times both the sides of (3.6) we have

\[
(3.7) \quad \sum_{k=r}^{n} \bar{P}_k(t) \bar{P}_k^{(r)}(x) = \theta_n \frac{\{ \bar{P}_{n+1}^{(r)}(t) \bar{P}_n(t) - \bar{P}_n^{(r)}(x) \bar{P}_{n+1}(t) \}}{x - t} + \\
\quad \quad + \theta_n \sum_{v=0}^{r-1} (-1)^{v-r} \frac{r!}{v!} \{ \bar{P}_{n+1}^{(v)}(t) \bar{P}_n(t) - \bar{P}_n^{(v)}(x) \bar{P}_{n+1}(t) \}. \quad (x - t)^{r-v+1}
\]

Then we have

\[
(3.8) \quad \int_{\delta_n(x)} (1 - t^2)^{(p+q)/2} \left| \sum_{k=r}^{n} \bar{P}_k(t) \bar{P}_k^{(r)}(x) \right| dt \leq \\
\leq \int_{\delta_n(x)} (1 - t^2)^{(p+q)/2} \frac{\{ \bar{P}_{n+1}^{(r)}(t) \bar{P}_n(t) - \bar{P}_n^{(r)}(x) \bar{P}_{n+1}(t) \}}{x - t} dt + \\
+ \int_{\delta_n(x)} (1 - t^2)^{(p+q)/2} \left| \sum_{v=0}^{r-1} (-1)^{v-r} \frac{r!}{v!} \{ \bar{P}_{n+1}^{(v)}(t) \bar{P}_n(t) - \bar{P}_n^{(v)}(x) \bar{P}_{n+1}(t) \} \right| dt = u_1 + u_2.
\]

Since \( |x - t| > 1/n \) for \( t \in \delta_n(x) \) therefore we have by using (2.3) and (2.6),

\[
(3.9) \quad u_1 \leq K_r n^{2r+1/2} \int_{\delta_n(x)} (1 - t^2)^{(p+q)/2} \left[ |\bar{P}_n(t)| + |\bar{P}_n(t)| \right] \frac{dt}{|x - t|} \leq \\
\leq K_r n^{2r+1/2} \int_{\delta_n(x)} \frac{dt}{|x - t|} \leq \log n, \quad x \in [-1, 1].
\]

For \( u_2 \) we have, on making use of (2.3) and (2.6),

\[
(3.10) \quad u_2 \leq \int_{\delta_n(x)} (1 - t^2)^{(p+q)/2} \sum_{v=0}^{r-1} (-1)^{v-r} \frac{r!}{v!} \{ \bar{P}_{n+1}^{(v)}(t) |\bar{P}_n(t)| + |\bar{P}_n^{(v)}(x) \bar{P}_{n+1}(t) | \} \frac{dt}{|x - t|^{r-v+1}} \\
\leq \lambda_r \sum_{v=0}^{r-1} n^{2v+1/2} \int_{\delta_n(x)} \frac{dt}{|x - t|^{r-v+1}} \leq \lambda_r \int_{\delta_n(x)} \frac{dt}{|x - t|^{r+1/2}} \leq \lambda_r n^{2r-1/2}, \quad x \in [-1, 1].
\]

Hence from (3.5), (3.8), (3.9) and (3.10) the lemma is obtained.

**Lemma 3.3.** Let \( f^{(q)}(x) \in \text{Lip } \alpha \) \( 0 < \alpha < 1 \) in \([-1, 1]\); then there is a polynomial \( Q_n(x) \) of degree at most \( n \) possessing the following properties:

\[
(3.11) \quad |f(x) - Q_n(x)| \leq \frac{c_{16}}{n^{q+2}} \left[ (1 - x^2)^{q+2} + \frac{1}{n^{q+2}} \right]
\]

352
and

\[
|f^{(r)}(x) - Q_n^{(r)}(x)| \leq \frac{\mu_r}{n^{p+\alpha-r}} \left[ (\sqrt{(1-x^2)})^{p+\alpha-r} + \frac{1}{n^{q+\alpha-r}} \right]
\]

uniformly in \([-1, 1]\) and \(r = 1, 2, \ldots, q\).

For \(r = 1\) the lemma has been proved by Saxena [7] and for \(r \geq 2\) it can be proved on the same lines.

4. The proof of Theorem. We shall confine ourselves to proving (1.7).

We write

\[
|f^{(r)}(x) - S_n^{(r)}(x)| = |f^{(r)}(x) - Q_n^{(r)}(x) + Q_n^{(r)}(x) - S_n^{(r)}(x)| \leq 
\]

\[
\leq |f^{(r)}(x) - Q_n^{(r)}(x)| + \int_{-1}^{1} |Q_n(t) - f(t)| \left| \sum_{k=r}^{n} \bar{P}_k(t) \bar{P}_k^{(r)}(x) \right| dt .
\]

Now using lemma 3.3 we have

\[
|f^{(r)}(x) - S_n^{(r)}(x)| \leq \frac{\mu_r}{n^{p+\alpha-r}} \left[ (\sqrt{(1-x^2)})^{p+\alpha-r} + \frac{1}{n^{q+\alpha-r}} \right] + 
\]

\[
+ \frac{c_{16}}{n^{p+\alpha}} \int_{-1}^{1} \left\{ (1 - t^2)^{(p+\alpha)/2} + \frac{1}{n^{p+\alpha}} \right\} \left| \sum_{k=r}^{n} \bar{P}_k(t) \bar{P}_k^{(r)}(x) \right| dt \leq 
\]

\[
\leq \frac{\mu_r'}{n^{p+\alpha-r}} + \frac{c_{16}}{n^{p+\alpha}} \int_{-1}^{1} (1 - t^2)^{(p+\alpha)/2} \left| \sum_{k=r}^{n} \bar{P}_k(t) \bar{P}_k^{(r)}(x) \right| dt + 
\]

\[
+ \frac{c_{16}}{n^{2(p+2\alpha)}} \int_{-1}^{1} \left| \sum_{k=r}^{n} \bar{P}_k(t) \bar{P}_k^{(r)}(x) \right| dt
\]

which, with the help of (3.4) and (3.2), yields

\[
|f^{(r)}(x) - S_n^{(r)}(x)| \leq \frac{\mu_r'}{n^{p+\alpha-r}} + \frac{c_{16}c_r^* \log n}{n^{p+\alpha-2r-1/2}} + \frac{c_{16}c_{12}}{n^{2p+2\alpha-2r-1}} \leq 
\]

\[
\leq \frac{c_r \log n}{n^{p+\alpha-2r-1/2}}, \quad p \geq 2r .
\]

This completes the proof of (1.7). The proof of (1.6) can be given in the same manner. One can easily see that if \(r = 0\) we have (1.2) and if \(r = 1\) we get (1.5).

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References


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