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REMARKS ON A THEOREM OF P. K. SUETIN

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1. Let

$$S_n(x) = \sum_{k=0}^{n} a_k \bar{P}_k(x) \tag{1.1}$$

denote the *nth* partial sum of the Fourier Legendre series of a function $f(x)$. It is well-known that $S_n(x)$ converges uniformly to $f(x)$ in $[-1, 1]$ if $f(x)$ has a continuous second derivative on $[-1, 1]$. Recently SUETIN [4] has shown that $S_n(x)$ converges uniformly to $f(x)$ if $f(x)$ belongs to a Lipschitz class of order greater than $1/2$ in $[-1, 1]$.

More precisely he has established the following result.

Theorem 1. (P. K. Suetin [4]). *If $f(x)$ has p continuous derivatives on $[-1, 1]$ and $f^{(p)}(x) \in \mathrm{Lip}\,\alpha$, then*

$$|f(x) - S_n(x)| \leqq \frac{c_1 \log n}{n^{p+\alpha-1/2}}, \quad x \in [-1, 1], \tag{1.2}$$

for $p + \alpha \geqq 1/2$.

In establishing this remarkable theorem he has employed the following well known theorem of A. F. TIMAN [6] which is a stronger form of Jackson's theorem.

Theorem 2. *If $f(x)$ has p continuous derivatives on $[-1, 1]$ and $f^{(p)}(x) \in \mathrm{Lip}\,\alpha$, then there is a sequence of polynomials $\{\varrho_n(x)\}$ for which*

$$|f(x) - \varrho_n(x)| \leqq \frac{c_2}{n^{p+\alpha}} \left(\sqrt{(1 - x^2)} + \frac{1}{n}\right)^{p+\alpha}, \quad x \in [-1, 1].$$

Very recently SAXENA [3] has proved the following theorem for $S_n'(x)$, the first derivative of $S_n(x)$ with respect to x.

Theorem 3 (R. B. Saxena [3]). *If $f(x)$ has p continuous derivatives on $[-1, 1]$ and $f^{(p)}(x) \in \operatorname{Lip} \alpha$, then together with (1.2) the following inequalities hold:*

$$(1.3) \qquad (1 - x^2)^{3/4} |f'(x) - S_n'(x)| \leqq \frac{c_3 \log n}{n^{p+\alpha-1}}, \quad (0 < \alpha < 1,\ p \geqq 1),$$

$$(1.4) \qquad (1 - x^2)^{1/2} |f'(x) - S_n'(x)| \leqq \frac{c_4 \log n}{n^{p+\alpha-3/2}}, \quad (\tfrac{1}{2} < \alpha < 1,\ p \geqq 1)$$

and

$$(1.5) \qquad |f'(x) - S_n'(x)| \leqq \frac{c_5 \log n}{n^{p+\alpha-5/2}}, \quad (\tfrac{1}{2} < \alpha < 1,\ p \geqq 2)$$

uniformly in $[-1, 1]$.

In connection with theorem 1 we shall prove the following theorem which generalizes theorem 3.

Theorem 4. *If $f(x)$ has p continuous derivatives on $[-1, 1]$ and $f^{(p)}(x) \in \operatorname{Lip} \alpha$, then together with (1.3) and (1.4) the following inequalities hold:*

$$(1.6) \qquad (1 - x^2)^{1/4} |f(x) - S_n(x)| \leqq \frac{c_6 \log n}{n^{p+\alpha}}, \quad (p + \alpha \geqq \tfrac{1}{2})$$

and

$$(1.7) \qquad |f^{(r)}(x) - S_n^{(r)}(x)| \leqq \frac{c_r \log n}{n^{p+\alpha-2r-1/2}}, \quad (p \geqq 2r,\ \tfrac{1}{2} < \alpha < 1)$$

uniformly in $[-1, 1]$.

2. To prove the theorem we shall need the following well-known results on Legendre polynomials. The orthonormalized Lengendre polynomial $\bar{P}_n(x)$ is given by [1]

$$(2.1) \qquad \bar{P}_n(x) = \sqrt{\left(\frac{n+1}{2}\right)}\, P_n(x),$$

where $P_n(x)$ denotes the *nth* Legendre polynomial with the normalization $P_n(1) = 1$. From [1], [2] and [5] we have for $-1 \leqq x \leqq 1$,

$$(2.2) \qquad |\bar{P}_n(x)| \leqq c_7 \sqrt{n}$$

and the inequality

$$(2.3) \qquad (1 - x^2)^{1/4} |\bar{P}_n(x)| \leqq c_8 .$$

For the derivatives of $\bar{P}_n(x)$ we have the following inequalities which hold for $-1 \leqq x \leqq 1$,

$$(2.4) \qquad (1 - x^2)^{1/2} \left|\bar{P}_n'(x)\right| \leqq c_9 n^{3/2} ,$$

$$(2.5) \qquad (1 - x^2)^{3/4} \left|\bar{P}_n'(x)\right| \leqq c_{10} n$$

and the Markov's inequality

$$(2.6) \qquad \left|\bar{P}_n^{(r)}(x)\right| \leqq c_{11} n^{2r+1/2} , \quad r = 0, 1, 2, \ldots$$

3. In order to prove Theorem 4 we require the following lemmas.

Lemma 3.1. *For $-1 \leqq x \leqq 1$, we have*

$$(3.1) \qquad (1 - x^2)^{1/4} \int_{-1}^{1} \left|\sum_{k=0}^{n} \bar{P}_k(t)\, \bar{P}_k(x)\right| \mathrm{d}t \leqq c_{11} n^{1/2}$$

and

$$(3.2) \qquad \int_{-1}^{1} \left|\sum_{k=r}^{n} \bar{P}_k(t)\, \bar{P}_k^{(r)}(x)\right| \mathrm{d}t \leqq c_{12} n^{2r+1} .$$

Proof. We give here the proof for (3.2) only. The proof for (3.1) can be given on the same lines. Making use of (2.6) we have

$$\int_{-1}^{1} \left[\sum_{k=r}^{n} \bar{P}_k(t)\, \bar{P}_k^{(r)}(x)\right]^2 \mathrm{d}t = \sum_{k=r}^{n} \left|\bar{P}_k^{(r)}(x)\right|^2 \leqq c_{13} \sum_{k=0}^{n} k^{4r+1} \leqq c_{14} n^{4r+2} ,$$

from which (3.2) follows.

Lemma 3.2. *We have for $-1 \leqq x \leqq 1$ and $\alpha \geqq 1/2$,*

$$(3.3) \qquad (1 - x^2)^{1/4} \int_{-1}^{1} (\sqrt{(1 - t^2)})^{p+\alpha} \left|\sum_{k=0}^{n} \bar{P}_k(t)\, \bar{P}_k(x)\right| \mathrm{d}t \leqq c_{15} \log n$$

and

$$(3.4) \qquad \int_{-1}^{1} (\sqrt{(1 - t^2)})^{p+\alpha} \left|\sum_{k=r}^{n} \bar{P}_k(t)\, \bar{P}_k^{(r)}(x)\right| \mathrm{d}t \leqq c_r^* n^{2r+1/2} \log n .$$

Proof. We shall prove (3.4) only and (3.3) can be proved in the same manner. Let us denote by $\Delta_n(x)$ the part of $[-1, 1]$ on which $|x - t| \leqq 1/n$ and by $\delta(x)$ the rest of the interval. Making use of (2.3) and (2.6), we obtain

$$(3.5) \qquad \int_{\Delta_n(x)} (1 - t^2)^{(p+\alpha)/2} \left|\sum_{k=r}^{n} \bar{P}_k(t)\, \bar{P}_k^{(r)}(x)\right| \mathrm{d}t \leqq$$

$$\leqq \int_{\Delta_n(x)} \sum_{k=r}^{n} (1 - t^2)^{(p+\alpha)/2} \left|\bar{P}_k(t)\right| \left|\bar{P}_k^{(r)}(x)\right| \mathrm{d}t \leqq K_r^* \frac{1}{n} \sum_{k=0}^{n} k^{2r+1/2} \leqq K_r n^{2r+1/2} .$$

To estimate the integral over $\delta_n(x)$ we make use of the Christoffel formula [5].

$$(3.6) \qquad \sum_{k=0}^{n} \bar{P}_k(t)\, \bar{P}_k(x) = \theta_n \frac{\bar{P}_{n+1}(x)\, \bar{P}_n(t) - \bar{P}_n(x)\, \bar{P}_{n+1}(t)}{x - t}, \quad 0 < \theta_n \leqq 1 .$$

On differentiating r times both the sides of (3.6) we have

$$(3.7) \qquad \sum_{k=r}^{n} \bar{P}_k(t)\, \bar{P}_k^{(r)}(x) = \theta_n \frac{\{\bar{P}_{n+1}^{(r)}(x)\, \bar{P}_n(t) - \bar{P}_n^{(r)}(x)\, \bar{P}_{n+1}(t)\}}{x - t} +$$

$$+ \theta_n \sum_{\nu=0}^{r-1} \frac{(-1)^{r-\nu}\, r!\, \{\bar{P}_{n+1}^{(\nu)}(x)\, \bar{P}_n(t) - \bar{P}_n^{(\nu)}(x)\, \bar{P}_{n+1}(t)\}}{\nu!\, (x - t)^{r-\nu+1}} .$$

Then we have

$$(3.8) \qquad \int_{\delta_n(x)} (1-t^2)^{(p+\alpha)/2} \left|\sum_{k=r}^{n} \bar{P}_k(t)\, \bar{P}_k^{(r)}(x)\right| \mathrm{d}t \leqq$$

$$\leqq \int_{\delta_n(x)} (1-t^2)^{(p+\alpha)/2} \left|\frac{\bar{P}_{n+1}^{(r)}(x)\, \bar{P}_n(t) - \bar{P}_n^{(r)}(x)\, \bar{P}_{n+1}(t)}{x - t}\right| \mathrm{d}t +$$

$$+ \int_{\delta_n(x)} (1-t^2)^{(p+\alpha)/2} \left|\sum_{\nu=0}^{r-1} \frac{(-1)^{r-\nu}\, r!\, \{\bar{P}_{n+1}^{(\nu)}(x)\, \bar{P}_n(t) - \bar{P}_n^{(\nu)}(x)\, \bar{P}_{n+1}(t)\}}{\nu!\, (x - t)^{r-\nu+1}}\right| \mathrm{d}t = u_1 + u_2 .$$

Since $|x - t| > 1/n$ for $t \in \delta_n(x)$ therefore we have by using (2.3) and (2.6),

$$(3.9) \qquad u_1 \leqq K'_r n^{2r+1/2} \int_{\delta_n(x)} (1-t^2)^{(p+\alpha)/2} \left[|\bar{P}_n(t)| + |\bar{P}_{n+1}(t)|\right] \frac{\mathrm{d}t}{|x - t|} \leqq$$

$$\leqq K''_r n^{2r+1/2} \int_{\delta_n(x)} \frac{\mathrm{d}t}{|x - t|} \leqq K'''_r n^{2r+1/2} \log n , \quad x \in [-1, 1] .$$

For u_2 we have, on making use of (2.3) and (2.6),

$$(3.10) \quad u_2 \leqq \int_{\delta_n(x)} (1-t^2)^{(p+\alpha)/2} \sum_{\nu=0}^{r-1} \frac{r!}{\nu!} \frac{\{|\bar{P}_{n+1}^{(\nu)}(x)|\, |\bar{P}_n(t)| + |\bar{P}_n^{(\nu)}(x)|\, |\bar{P}_{n+1}(t)|\}}{|x - t|^{r-\nu+1}} \mathrm{d}t \leqq$$

$$\leqq \lambda_r \sum_{\nu=0}^{r-1} n^{2\nu+1/2} \int_{\delta_n(x)} \frac{\mathrm{d}t}{|x - t|^{r-\nu+1}} \leqq \lambda'_r \sum_{\nu=0}^{r-1} n^{r+\nu+1/2} \leqq \lambda''_r n^{2r-1/2} , \quad x \in [-1, 1] .$$

Hence from (3.5), (3.8), (3.9) and (3.10) the lemma is obtained.

Lemma 3.3. *Let $f^{(q)}(x) \in \mathrm{Lip}\, \alpha$ $(0 < \alpha < 1)$ in $[-1, 1]$; then there is a polynomial $Q_n(x)$ of degree at most n possessing the following properties:*

$$(3.11) \qquad |f(x) - Q_n(x)| \leqq \frac{c_{16}}{n^{q+\alpha}} \left[(\sqrt{(1 - x^2)})^{q+\alpha} + \frac{1}{n^{q+\alpha}}\right]$$

and

$$(3.12) \qquad |f^{(r)}(x) - Q_n^{(r)}(x)| \leqq \frac{\mu_r}{n^{q+\alpha-r}}\left[(\sqrt{(1-x^2)})^{q+\alpha-r} + \frac{1}{n^{q+\alpha-r}}\right]$$

uniformly in $[-1, 1]$ *and* $r = 1, 2, \ldots, q$.

For $r = 1$ the lemma has been proved by Saxena [7] and for $r \geqq 2$ it can be proved on the same lines.

4. The proof of Theorem. We shall confine ourselves to proving (1.7).

We write

$$(4.1) \qquad |f^{(r)}(x) - S_n^{(r)}(x)| = |f^{(r)}(x) - Q_n^{(r)}(x) + Q_n^{(r)}(x) - S_n^{(r)}(x)| \leqq$$

$$\leqq |f^{(r)}(x) - Q_n^{(r)}(x)| + \int_{-1}^{1} |Q_n(t) - f(t)| \left|\sum_{k=r}^{n} \bar{P}_k(t)\, \bar{P}_k^{(r)}(x)\right| dt\,.$$

Now using lemma 3.3 we have

$$|f^{(r)}(x) - S_n^{(r)}(x)| \leqq \frac{\mu_r}{n^{p+\alpha-r}}\left[(\sqrt{(1-x^2)})^{p+\alpha-r} + \frac{1}{n^{p+\alpha-r}}\right] +$$

$$+ \frac{c_{16}}{n^{p+\alpha}} \int_{-1}^{1} \left\{(1-t^2)^{(p+\alpha)/2} + \frac{1}{n^{p+\alpha}}\right\} \left|\sum_{k=r}^{n} \bar{P}_k(t)\, \bar{P}_k^{(r)}(x)\right| dt \leqq$$

$$\leqq \frac{\mu_r'}{n^{p+\alpha-r}} + \frac{c_{16}}{n^{p+\alpha}} \int_{-1}^{1} (1-t^2)^{(p+\alpha)/2} \left|\sum_{k=r}^{n} \bar{P}_k(t)\, \bar{P}_k^{(r)}(x)\right| dt +$$

$$+ \frac{c_{16}}{n^{2p+2\alpha}} \int_{-1}^{1} \left|\sum_{k=r}^{n} \bar{P}_k(t)\, \bar{P}_k^{(r)}(x)\right| dt$$

which, with the help of (3.4) and (3.2), yields

$$|f^{(r)}(x) - S_n^{(r)}(x)| \leqq \frac{\mu_r'}{n^{p+\alpha-r}} + \frac{c_{16}c_r^* \log n}{n^{p+\alpha-2r-1/2}} + \frac{c_{16}c_{12}}{n^{2p+2\alpha-2r-1}} \leqq$$

$$\leqq \frac{c_r \log n}{n^{p+\alpha-2r-1/2}}, \quad p \geqq 2r\,.$$

This completes the proof of (1.7). The proof of (1.6) can be given in the same manner. One can easily see that if $r = 0$ we have (1.2) and if $r = 1$ we get (1.5).

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References

[1] *I. P. Natanson:* Constructive theory of functions (in Russian: GITTL, Moscow, 1949; English transl.: FUPC, New York, 1964).
[2] *G. Sansone:* Orthogonal functions (in Italian; English transl., IPI, New York, 1959).
[3] *R. B. Saxena:* Expansion of continuous differentiable functions in Fourier Legendre Series, Canadian J. of Math., *19* (1967), 823—827.
[4] *P. K. Suetin:* Representation of continuous and differentiable function by Fourier series of Legendre polynomials, Soviet Math. Dokl., *5* (1964), 1408—1410.
[5] *G. Szegö:* Orthogonal polynomials, Amer. Math. Soc. Colloq. Pub., 2nd ed. (1959).
[6] *A. F. Timan:* Theory of approximation of functions of a real variable (English transl.: Fizmatgiz, Moscow, 1960).
[7] *R. B. Saxena:* On mixed type lacunary interpolation II, Acta Math. Acad. Sci. Hung. *14* (1963), 1—19.

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