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# ON UNISOLVENT SYSTEMS*)

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The purpose of this paper is to give some miscellaneous results concerning unisolvent systems of equations.

Let $f_1, f_2, \ldots, f_n$ be real valued functions defined on a set $S$. Denote by $|f_i(x_j)|_{i,j=1}^n$ the $n$ by $n$ determinant

$$\begin{vmatrix} f_1(x_1) & f_1(x_2) & \ldots & f_1(x_n) \\ f_2(x_1) & f_2(x_2) & \ldots & f_2(x_n) \\ \vdots & & & \\ f_n(x_1) & f_n(x_2) & \ldots & f_n(x_n) \end{vmatrix}$$

where $x_1, x_2, \ldots, x_n$ is a subset of $S$ consisting of $n$ distinct points.

**Definition.** The system of $n$ functions $f_1, f_2, \ldots, f_n$ is *unisolvent* on the set $S$ if and only if $|f_i(x_j)|_{i,j=1}^n \neq 0$ for every selection of $n$ distinct points in $S$ [1, page 31].

**Theorem 1.** *Let $f_1, f_2, \ldots, f_n$ be an even number of continuous functions which are unisolvent on the closed interval $[a, b]$.*

*Suppose that $f_{n+1}$ is continuous on the open interval $(a, b)$ with $\lim_{x\to a} f_{n+1}(x) = -\infty$ and $\lim_{x\to b} f(x) = +\infty$. Then the set $f_1, f_2, \ldots, f_{n+1}$ cannot be unisolvent on $(a, b)$.*

Proof. Let the numbers $c$ and $d$ be chosen such that $a < c < d < b$. Then $f_{n+1}$ is bounded on $[c, d]$ and the $f_i$'s, $i = 1, 2, \ldots, n$ are bounded on $[a, b]$. Let $M$ be a number which is greater than the absolute value of all of these upper and lower bounds. Since the expansion of $|f_i(x_j)|_{i,j=1}^n$ contains $n!$ terms, with $n$ factor in each term, it follows that an upper bound for the absolute value of $|f_i(x_j)|_{i,j=1}^n$ is $n!\, M^n$. Choose $n + 1$ points such that $c \leqq x_1 < x_2 < \ldots < x_n < x_{n+1} \leqq d$ and consider $|f_i(x_j)|_{i,j=1}^{n+1}$. Denote the cofactor of $f_i(x_j)$ by $F_i(x_j)$ and we have

$$|f_i(x_j)|_{i,j=1}^{n+1} = f_{n+1}(x_1)\, F_{n+1}(x_1) + \ldots + f_{n+1}(x_{n+1})\, F_{n+1}(x_{n+1})$$

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If we hold $x_2, x_3, \ldots, x_{n+1}$ fixed and let $x_1$ tend toward $a$, the sign of $|f_i(x_j)|_{i,j=1}^{n+1}$ will be dominated by the sign of $f_{n+1}(x_1)\, F_{n+1}(x_1)$. To see this, note that (1) $F_{n+1}(x_1)$ is a constant, since it does not contain $x_1$, (2) $F_{n+1}(x_1)$ is not zero, since $f_1, \ldots, f_n$ are unisolvent and (3) $F_{n+1}(x_i)$, $i = 2, \ldots, n$ is bounded in absolute value by $n!\, M^n$.

On the other hand, if we hold $x_1, x_2, \ldots, x_n$ fixed and let $x_{n+1}$ tend toward $b$, the sign of $|f_i(x_j)|_{i,j=1}^{n+1}$ is dominated by $f_{n+1}(x_{n+1})\, F_{n+1}(x_{n+1})$. Now $F_{n+1}(x_1)$ has the same sign as $F_{n+1}(x_{n+1})$ since the $f_1, f_2, \ldots, f_n$ are continuous and unisolvent and the deminsions of $|f_1(x_j)|_{i,j=1}^{n+1}$ is odd. Since the determinant $|f_i(x_j)|_{i,j=1}^{n+1}$ takes on values continuously, and $f_{n+1}(x_1)$ and $f_{n+1}(x_n)$ have opposite signs in the limits above, it follows that $|f_i(x_j)|_{i,j=1}^{n+1}$ must be zero for some value of $x_1$ or $x_{n+1}$. Therefore $f_1, \ldots$ $\ldots, f_{n+1}$ cannot be unisolvent on $(a, b)$.

**Theorem 2.** *Let $f_1, f_2, \ldots, f_n$ be an odd number of continuous functions which are unisolvent on the closed interval $[a, b]$, Suppose that $f_{n+1}$ is continuous on the open interval $(a, b)$ with*

$$\lim_{x \to a} f_{n+1}(x) = \lim_{x \to b} f_{n+1}(x) = \infty\,.$$

*Then $f_1, f_2, \ldots, f_{n+1}$ cannot be unisolvent on $(a, b)$.*

The proof is similar to that of Theorem 1.

**Theorem 3.** *Let $f_1, \ldots, f_n$ be $n$ functions defined on a set $S$. If any $n - k$, $0 \leqq k \leqq$ $\leqq n - 2$, of these functions have common values for $k + 2$ points in $S$, then $f_1, \ldots, f_n$ are not unisolvent.*

Proof. We may assume that $f_1, \ldots, f_{n-k}$ have common values at the points $x_1, x_2, \ldots, x_{k+2}$. Choose any other $n - (k + 2)$ points of $S$ and expand $|f_i(x_j)|_{i,j=1}^{n}$ by minors with respect to the last row. After $k$ expansions, we have for a first term

$$f_n(x_1)\, f_{n-1}(x_2) \ldots f_{n-k+1}(x_k) \begin{vmatrix} f_1(x_{k+1}) & \ldots & f_1(x_n) \\ \vdots & & \\ f_{n-k}(x_{k+1}) & \ldots & f_{n-k}(x_n) \end{vmatrix}.$$

But the first two columns of this determinant are identical. Thus this term is 0. A similar argument hold for each term. Therefore $f_1, \ldots, f_n$ are not unisolvent.

*References*

[1] *Davis, Philip J.:* Interpolation and Approximation, Blaisdell Publishing Company, New York, 1963.

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