

Václav Chvátal

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SOME RELATIONS AMONG INVARIANTS OF GRAPHS

VÁCLAV CHVÁTAL, Waterloo

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1. INTRODUCTION

R. E. NETTLETON [2] investigated relations among number of vertices, connectivity, diameter, degrees of vertices and chromatic number of graphs. We shall use or improve his results to present a complete survey of the best possible bounds for the last two invariants in terms of the first three.

Our notation follows HARARY [1]. Particularly, we reserve: a letter p for a number of points, κ for connectivity, d for diameter, χ for chromatic number of a graph and $d(v)$ for a degree of a point v .

2. RESULTS

Nettleton has proved among others the following theorems:

- N. 1. $\chi \geq p/(p - \kappa)$.
- N. 2. *If for any positive integer j , $d \geq 2j + 1$, then $\chi \leq p - 2(\kappa j - \kappa + 1)$.*
- N. 3. *If $d = j + 1$ for $j \geq 3$, no point of G can have degree $> p - \kappa(j - 3) - 3$.*
- N. 4. $d - 4 \leq (p - \kappa - 3)/\kappa$.

Later on, M. E. WATKINS [3] improved N. 4. into the following best possible form:
 $p \geq \kappa(d - 1) + 2$.

We shall prove two theorems:

Theorem 1.

$$(1a) \quad p/(p - \kappa) \leq \chi \leq p - 1 \quad \text{if } d = 2$$

$$(1b) \quad 2 \leq \chi \leq p - \kappa(d - 3) - 2 \quad \text{if } 3 \leq d < \infty$$

and these bounds are the best possible.

Theorem 2. Given any point v one has

- (2a) $\kappa \leq d(v) \leq p - 1$ if $d = 2$,
 (2b) $\kappa \leq d(v) \leq p - 2$ if $d = 3$,
 (2c) $\kappa \leq d(v) \leq p - \kappa(d - 4) - 3$ if $4 \leq d < \infty$

and these bounds are the best possible.

These theorems show that N. 1. and N. 3. are best possible while N. 2. is not.

3. SOME SPECIAL GRAPHS

In this paragraph, we shall construct graphs employed as examples in proofs.

1. $G_1(p, \kappa)$ where $\kappa \leq p - 2$ is a complete $(a + 1)$ -partite graph $K(p - \kappa, p - \kappa, \dots, p - \kappa, b)$ where $p = a(p - \kappa) + b$, $0 \leq b < p - \kappa$. It is easy to see that G_1 is κ -connected, its diameter is 2 and its chromatic number a (if $b = 0$), resp. $a + 1$ (if $b > 0$). In both cases, the last quantity is the least integer $\geq p/(p - \kappa)$.

2. $G_2(p, \kappa)$ where $\kappa \leq p - 2$ has a set of points $V_1 \cup V_2 \cup \{w\}$, $V_1, V_2, \{w\}$ are pairwise disjoint, $|V_1| = p - \kappa - 1$, $|V_2| = \kappa$. A couple uv is not a line of G_2 iff $u \in V_1, v = w$. G_2 is κ -connected, its diameter is 2 and its chromatic number is $p - 1$.

3. $G_3(p, \kappa, d)$ where $d \geq 3, \kappa(d - 1) \leq p - 2$ has a set of points $V = V_0 \cup V_1 \cup \dots \cup V_d$ where $|V_0| = |V_d| = 1, |V_1| = p - \kappa(d - 2) - 2, |V_i| = \kappa$ for $i = 2, 3, \dots, d - 1, V_i \cap V_j = \emptyset$ for $i \neq j$. A couple uv is a line of G_3 iff $u \in V_i, v \in V_j, |i - j| = 1$. G_3 is κ -connected, has a diameter d and a chromatic number 2.

4. $G_4(p, \kappa, d)$ where $d \geq 3, \kappa(d - 1) \leq p - 2$ has the same set of points as G_3 . A couple uv is a line of G_4 iff $u \in V_j, v \in V_i, |u - v| \leq 1$. G_4 is κ -connected, has a diameter d and a chromatic number $p - \kappa(d - 3) - 2$.

4. PROOFS

Proof of Theorem 1: The left side of (1a) is Nettleton's result N. 1., the right side of (1a) and the left side of (1b) are trivial. Now, let G be a graph with set of points V and a diameter $d \geq 3$. There are $u, v \in V$ such that $d(u, v) = d$. Put $V_i = \{w \in V; d(u, w) = i\}$ for $i = 0, 1, \dots, d$. A graph $G - V_i$ is disconnected whenever $1 \leq i < d$. Hence, $|V_i| \geq \kappa$ and one can find $\{v_i^1, v_i^2, \dots, v_i^\kappa\} \subset V_i (1 \leq i \leq d)$. Write v_0^1 for u and v_d^1 for v . Remark that xy is not a line if $x \in V_i, y \in V_j, |i - j| \geq 2$ and define

color classes $O_1, O_2, \dots, O_\kappa, E_1, E_2, \dots, E_\kappa$ by

$$O_j = \{v_i^j; i \text{ odd}\}, \quad E_j = \{v_i^j; i \text{ even}\}.$$

Consider each of the remaining $p - (\kappa(d - 1) + 2)$ vertices as a color class consisting of a single point. Altogether, $2\kappa + p - (\kappa(d - 1) + 2) = p - \kappa(d - 3) + 2$ color classes were introduced and (1b) is proved.

To show that these bounds are best possible remember the result of Watkins and use the graphs $G_1(p, \kappa)$, $G_2(p, \kappa)$, $G_3(p, \kappa, d)$ and $G_4(p, \kappa, d)$ as examples.

Proof of Theorem 2: The left sides are trivial (cf. [1], Theorem 5.1) and so is the right side of (2a). The right side of (2b) follows by the fact that $d(v) = p - 1$ implies $d \leq 2$. The right side of (2c) is Nettleton's result N. 3.

To see that the bounds are best possible consider graphs $G_2(p, \kappa)$ and $G_4(p, \kappa, d)$.

References

- [1] *F. Harary*: Graph Theory, Addison-Wesley, Reading, Mass. 1969.
- [2] *R. E. Nettleton*: Some Generalized Theorems on Connectivity, *Canad. J. Math.* *XII* (1960), 546—554.
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Author's address: University of Waterloo, Waterloo, Ontario, Canada.