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LIMITS OF APPROXIMATELY CONTINUOUS FUNCTIONS

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In the paper [1] it is proved that any function of the second class of Baire is the limit of a sequence of derivatives. But it does not follow from this proof that any such function is the limit of a sequence of bounded derivatives. In this paper it is proved that any function of the second class of Baire is the limit of a sequence of approximately continuous functions, consequently any such function is the limit of a sequence of bounded derivatives.

We denote by $R$ the real line and if $M \subset R$ we denote by $c_M$ the characteristic function of $M$.

At first we prove the following.

**Lemma.** Let $M$ be a $G_\delta$ and $F_\sigma$ set. Let $H$ be a $G_\delta$ set of measure zero. Let $H$ contain all points of $M$ which are not points of density of $M$ and all points of $R - M$ which are not points of density of $R - M$. Let $G$ be an open set, $G \supset H$.

Then there exists an approximately continuous function $\varphi$ such that $\{x \in R, \varphi(x) \neq c_M(x)\} \subset G - H$.

**Proof.** We put

$$E_1 = M \cap [H \cup (R - G)], \quad E_2 = (R - M) \cap [H \cup (R - G)],$$

$$N = R - (E_1 \cup E_2).$$

Then $E_1, N, E_2$ are disjoint sets, $E_1 \cup N \cup E_2 = R$ and $E_1, E_2$ are $G_\delta$ sets. Further

$$E_1 \cup N = R - E_2 = M \cup [(R - H) \cap G],$$

$$E_2 \cup N = R - E_1 = (R - M) \cup [(R - H) \cap G].$$

Hence it follows that $E_1 \cup N, E_2 \cup N$ are sets of the class $M_5$ (see [2]).

This implies that there exists an approximately continuous function $\varphi$ such that

$$\varphi(x) = 0 \quad \text{for} \quad x \in E_2$$

$$0 < \varphi(x) < 1 \quad \text{for} \quad x \in N$$

$$\varphi(x) = 1 \quad \text{for} \quad x \in E_1$$

(see lemma 12 in [2]).

This function $\varphi$ clearly satisfies the statement of the lemma.
**Theorem.** A function $f$ (possibly infinite) defined on $R$ is an element of the second class of Baire if and only if $f$ is the limit of a sequence of approximately continuous functions.

**Proof.** From the fact that every approximately continuous function is of the first class of Baire it follows that if $f$ is the limit of a sequence of approximately continuous functions then $f$ is an element of the second class of Baire.

Now let $f$ be an element of the second class of Baire. Then there exists a sequence $\{g_n\}_{n=1}^{\infty}$ of functions of the first class of Baire such that

$$\lim_{n \to \infty} g_n = f$$

and

$$g_n = \sum_{k=1}^{m_n} c_{k,n} h_{k,n}$$

where $c_{k,n}$ are real numbers and $h_{k,n}$ is the characteristic function of a set $H_{k,n}$ which is at the same time $G_\delta$ and $F_\sigma$ set (see [3]).

Let $H^*_n$ be the set of all points of $H_{k,n}$ which are not points of density of the set $H_{k,n}$ and all points of $R - H_{k,n}$ which are not points of density of $R - H_{k,n}$. We put

$$H^* = \bigcup_{k,n} H^*_n.$$  

Then $H^*$ is a set of measure zero. Let $H \supseteq H^*$ be a $G_\delta$ set of measure zero. Let $G_i$ be open sets such that

$$G_i \supseteq G_{i+1}, \quad \bigcap_{i=1}^{\infty} G_i = H.$$

According to the lemma there exist approximately continuous functions $\varphi_{k,n}$ such that

$$\{x \in R, \varphi_{k,n}(x) = h_{k,n}(x)\} \subseteq G_n - H.$$

We put $f_n = \sum_{k=1}^{m_n} c_{k,n} \varphi_{k,n}$.

The functions $f_n$ are clearly approximately continuous and the sequence $\{f_n\}_{n=1}^{\infty}$ converges to $f$.

**References**


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