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WEAK PRODUCT DECOMPOSITIONS OF DISCRETE LATTICES

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INTRODUCTION

Let \((A_i)_{i \in I}\) be a family of algebras. A weak product of \((A_i)_{i \in I}\) is a subalgebra \(B\) of the complete direct product \(A = \prod_{i \in I} A_i\) satisfying the following conditions: (1) two elements in \(B\) differ only in a finite number of components; and (2) if an element \(a \in A\) differs only in a finite number of components from an element \(b \in B\), then \(a \in B\) (cf. Grätzer [2]). In [6] there were investigated weak product decompositions of universal algebras with pairwise permutable congruence relations. The aim of this Note is to prove that any discrete lattice is a weak product of directly indecomposable factors. This result is then applied for studying isomorphisms of unoriented graphs of modular lattices; there is obtained a generalization of a theorem of [4].

In §1 there is defined the concept of a full subdirect product of lattices and it is proved that any two full subdirect decompositions of a lattice \(L\) have a common refinement. In §2 it is shown that any full subdirect decomposition of a discrete lattice \(L\) is a weak product decomposition of \(L\) and there is constructed the (uniquely determined) weak product decomposition of a discrete lattice \(L\) in which all factors are directly indecomposable. The isomorphisms of graphs of discrete modular lattices are studied in §3.

The notions of the weak product and of the full subdirect product can be defined for relational systems as well; in a forthcoming paper weak products of partially ordered sets will be investigated.

1. FULL SUBDIRECT DECOMPOSITIONS

The symbols \(\land, \lor\) or \(\cap, \cup\) denote lattice operations and set-theoretical operations, respectively. \(A \setminus B\) is the set of all elements of \(A\) that do not belong to \(B\). If \(L\) is a lattice, \(a, b \in L\), \(a \leq b\), then the interval \([a, b]\) is the set of all \(x \in L\) with the property \(a \leq x \leq b\). The interval \([a, b]\) is prime, if \(\text{card } [a, b] = 2\). A lattice \(L\) is said to be discrete, if all bounded chains in \(L\) are finite.
Let \( \{S_i : i \in I\} \) be a system of lattices. The complete direct product \( S = \Pi S_i (i \in I) \) is the set of all mappings \( f : I \to \bigcup S_i \) such that \( f(i) \in S_i \) for each \( i \in I \) with the partial order defined component-wise (i.e., \( f \leq g \) if \( f(i) \leq g(i) \) for each \( i \in I \)). When \( I = \{1, \ldots, n\} \), then \( S \) is denoted also by \( S = S_1 \times \cdots \times S_n \) and \( f(i) \) is the \( i \)-th component of the element \( f \).

Assume that \( L \) is a lattice and that there is an isomorphism \( \varphi \) of \( L \) into \( S \). Let \( x_0 \) be a fixed element of \( L, i \in I \). Denote

\[
A_i(x_0) = \{x \in L : \varphi(x)(j) = \varphi(x_0)(j) \text{ for each } j \in I, \ j \neq i\},
\]

\[
A_i^*(x_0) = \{x \in L : \varphi(x)(i) = \varphi(x_0)(i)\}.
\]

Clearly \( A_i(x_0) \) is a convex sublattice of \( L \) (the convexity means that for any \( x \in L \) and \( a_1, a_2 \in A_i(x_0) \), \( a_1 \leq x \leq a_2 \) implies \( x \in A_i(x) \)). Analogously, \( A_i^*(x_0) \) is a convex sublattice of \( L \) and

\[
(1) \quad \text{card } [A_i(x_0) \cap A_i^*(x)] \leq 1
\]

for any \( x \in L \). The isomorphism \( \varphi \) is said to determine a full subdirect decomposition of \( L \), whenever the following conditions (a) and (b) are satisfied:

(a) for any \( i \in I \) and any \( a^i \in A_i \) there is \( x \in L \) such that \( \varphi(x)(i) = a^i \);

(b) for any \( i \in I \) and any \( x, y \in L \) there exists \( z \in L \) such that

\[
\varphi(z)(i) = \varphi(x)(i) \quad \text{and} \quad \varphi(z)(j) = \varphi(y)(j) \quad \text{for any } j \in I, \ j \neq i.
\]

In the whole §1 we assume that \( \varphi \) satisfies (a) and (b). Obviously the element \( z \) satisfying (b) belongs to the set \( A_i^*(x) \cap A_i(y) \) and conversely, if \( z \) belongs to this set, then \( z \) fulfills (b). Hence, according to (1), the condition (b) is equivalent to

\[
(b') \quad \text{card } [A_i(x_0) \cap A_i^*(x)] = 1 \text{ for any } i \in I, \ x_0 \in L \text{ and any } x \in L.
\]

The elements of the one-element sets \( A_i(x_0) \cap A_i^*(x) \) and \( A_i(x) \cap A_i^*(x_0) \) will be denoted by \( x(A_i(x_0)) \) and \( x(A_i^*(x_0)) \), respectively. It is easy to verify that the following assertion holds true:

**1.1.** Let \( i \in I \). If \( x \in A_i(x_0) \) then \( x(A_i(x_0)) = x \).

For the sake of brevity denote \( A_i(x_0) = A_i^0 \).

Suppose that there is given another isomorphism \( \varphi' \) of \( L \) into \( \Pi B_k \) \( (k \in K) \) determining a full subdirect decomposition of \( L \). By analogical denotations let us put \( B_i(x_0) = B_i^0 \).
1.2. Let \( x \in L, x \geq x_0 \). Then \( x = \bigvee x(A_i^0) \) (\( i \in I \)).

Proof. Let \( i \in I, x(A_i^0) = y \). According to 1.1 \( y(A_j^0) = y = x(A_i^0) \) and for any \( j \in I, j \neq i \) we have \( y(A_j^0) = x_0 = x_0(A_j^0) \leq x(A_i^0) \), hence \( y \leq x \). Let \( z \in L, z \geq x(A_i^0) \) for each \( i \in I \). Then \( z(A_i^0) \geq (x(A_i^0)) (A_i^0) = x(A_i^0) \) for each \( i \in I \), thus \( z \geq x \).

1.3. Let \( i \in I, k \in K, x \in L, x \geq x_0 \). Then \( x(A_i^0) (B_k^0) = x(A_i^0) \land x(B_k^0) \).

Proof. Denote \( x(A_i^0) = u, x(B_k^0) = v, x(A_i^0) (B_k^0) = t \). Obviously \( x_0 \leq u \land v = w \), whence \( w \in [x_0, u] \cap [x_0, v] \subset A_i^0 \cap B_k^0 \). Consider the components of elements \( t, w \) with respect to \( B_k^0 \) and \( B_1^0 (l \in K, l \neq k) \); we get

\[
\begin{align*}
w(B_k^0) & \leq u(B_k^0) = t = t(B_k^0), \\
w(B_1^0) & = x_0 = t(B_1^0);
\end{align*}
\]

thus \( w = t \).

As a corollary, we obtain:

1.4. Let \( i \in I, k \in K, x \in L, x \geq x_0 \). Then \( x(A_i^0) (B_k^0) = x(B_k^0) (A_i^0) \).

In a dual way we can prove the assertions of the lemma 1.3 for the case \( x \leq x_0 \).

1.5. Let \( i \in I, k \in K, x \in L \). Then \( x(A_i^0) (B_k^0) \in A_i^0 \).

Proof. Put \( u = x_0 \land x, v = x_0 \lor x \). According to 1.3 we have \( u(A_i^0) (B_k^0) \in [x_0, v(A_i^0)] \subset A_i^0 \). Analogously, the dual of 1.3 gives \( u(A_i^0) (B_k^0) \in A_i^0 \). Since \( x(A_i^0) (B_k^0) \in [u(A_i^0) (B_k^0), v(A_i^0) (B_k^0)] \), from the convexity of \( A_i^0 \) it follows \( x(A_i^0) \).

For each \( A_i \) there correspond two congruence relations \( R(A_i) = R_i \) and \( R'(A_i) = R_i' \) defined as follows:

If \( x, y \in L \) and \( i \in I \), then we set \( x \equiv y(R_i) (x \equiv y(R_i')) \) if \( x \in A_i(y) (x \in A_i'(y)) \).

For \( k \in K \) let \( R_k \), \( R_k' \) have an analogical meaning. \( R_i \) and \( R_i' \) are permutative and \( R_i \land R_i' \) is the least congruence relation on \( L \). Let \( x_i \in L \) and denote \( A_i^1 = A_i(x_i), B_k^1 = B_k(x_i) \). Then for any \( z \in L \)

(2) \[
\begin{align*}
z \equiv z(A_i^0) (A_i^1) (R_i), \\
z(B_k^0) \equiv z(B_k^1) (R_k).
\end{align*}
\]

1.6. Let \( i \in I, k \in K, x \in L \). Then \( x(A_i^0) (B_k^0) = x(B_k^0) (A_i^0) \).

Proof. Put \( x_1 = x \land x_0 \) and denote

\[
\begin{align*}
u & = x(A_i^0) (B_k^0), \\
u & = x(B_k^0) (A_i^0), \\
u_1 & = x(A_i^1) (B_k^1), \\
u_1 & = x(B_k^1) (A_i^1).
\end{align*}
\]

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According to 1.5, $u, v \in A_i^0 \cap B_k^0$, hence $u \equiv v(R_i \land R_k)$. By (2)

$$u \equiv x(A_i^0)(R_k^0), \quad x(A_i^0) \equiv x(A_i^1)(R_i^0), \quad x(A_i^1) \equiv u_1(R_k^0),$$

thus $u \equiv u_1(R_i^0 \lor R_k^0)$ and analogously $v \equiv v_1(R_i^0 \lor R_k^0)$. From 1.4 we get $u_1 = v_1$, therefore $u \equiv v(R_i^0 \lor R_k^0)$. This implies $u \land v = u \lor v \lor v(R_i^0 \lor R_k^0)$, hence there are elements $u \land u \equiv v(R_i^0 \lor R_k^0)$ or $t_{m-1} \equiv t_m(R_k^0)$ or $t_{m-1} \equiv t_m(R_k^0)$. In the same time $u \equiv v(R_i^0 \lor R_k^0)$, whence $t_{m-1} = t_m$; therefore $u = v$.

1.6.1. Remark. The assertion of the lemma 1.6 could be deduced also from [3], Theorem 1, where a more general situation (concerning connected partially ordered sets) is dealt with; in the case of lattices, the present proof seems to be simpler.

For any $i \in I, k \in K$ denote $A_i^0 \cap B_k^0 = C_{ik}^0$ and let $\chi$ be a mapping of $L$ into $\Pi C_{ik}^0$ $(i \in I, k \in K)$, such that $\chi(x)(i, j) = x(A_i^0)(B_j^0)$ for any $x \in L$.

1.7. $\chi$ is an isomorphism of the lattice $L$ into $\Pi C_{ik}^0$ $(i \in I, k \in K)$.

Proof. Since the mappings $x \rightarrow x(A_i^0)$, $x \rightarrow x(B_k^0)$ are homomorphisms, $\chi$ is a homomorphism as well. We have to verify that $\chi$ is one-to-one. Let $x, y \in L, \chi(x) = \chi(y), i \in I$. Then $x(A_i^0)(B_k^0) = y(A_i^0)(B_k^0)$ for each $k \in K$, thus $x(A_i^0) = y(A_i^0)$. Since this holds for each $i \in I$, we get $x = y$.

1.8. Let $x, y \in L, i \in I, k \in K$. There exists $z \in L$ such that

$$z(C_{ik}^0) = x(C_{ik}^0),$$

$$z(C_{jl}^0) = y(C_{jl}^0) \quad \text{for each} \quad (j, l) \in I \times K, \quad (j, l) \neq (i, k).$$

Proof. Let us consider at first the elements $x(A_i^0)(B_k^0)$ and $y(A_i^0)$. Since the mapping $\varphi'$ satisfies (b), there exists $t \in L$ such that

$$t(B_k^0) = [x(A_i^0)(B_k^0)](B_k^0) = x(A_i^0)(B_k^0),$$

$$t(B_k^0) = y(A_i^0)(B_k^0) \quad \text{for each} \quad l \in K, \quad l \neq k.$$

Further consider the pair $t, y$. Since $\varphi$ satisfies (b), there is $z \in L$ such that

$$z(A_i^0) = t(A_i^0),$$

$$z(A_j^0) = y(A_j^0) \quad \text{for each} \quad j \in I, \quad j \neq i.$$

The element $z$ satisfies

$$z(A_i^0)(B_k^0) = t(A_i^0)(B_k^0) = t(B_k^0)(A_i^0) = x(A_i^0)(B_k^0)(A_i^0) = x(A_i^0)(B_k^0).$$

For any $l \in K, l \neq k$

$$z(A_i^0)(B_l^0) = t(A_i^0)(B_l^0) = t(B_l^0)(A_i^0) = y(A_i^0)(B_l^0).$$

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and for any \( j \in I, j \neq i \) and any \( s \in K \)

\[
z(A_i^0) (B_k^0) = y(A_i^0) (B_k^0).
\]

The proof is complete.

For any \( x \in C_{ik}^0 \) we have \( x(C_{ik}^0) = x(A_i^0) (B_k^0) = x \). From this and from 1.7 and 1.8 it follows:

1.9. The isomorphism \( \chi \) determines a full subdirect decomposition of the lattice \( L \).

The full subdirect decomposition of \( L \) with factors \( C_{ik}^0 \) determined by \( \chi \) is a refinement of both full subdirect decompositions with factors \( A_i^0 (i \in I) \) and \( B_k^0 (k \in K) \), respectively, in the following sense:

1.10. Let \( i \in I \). Then \( A_i^0 \) is a full subdirect product of lattices \( C_{ik}^0 \) (the isomorphism of \( A_i^0 \) into \( \Pi C_{ik}^0 (k \in K) \) being determined by the partial mapping \( \chi_{A_i^0} \)).

Proof. It suffices to verify that the mapping \( \chi \) possesses the property (b). Let \( x, y \in A_i^0 \) and let \( z \) have the same meaning as in 1.8. We have to show that \( z \) belongs to \( A_i^0 \). Denote \( u = x \wedge y \wedge x_0, v = x \vee y \vee x_0 \). Then all elements that were used in the proof of 1.8 belong to the interval \([u, v]\) and \([u, v] \subset A_i^0 \); therefore \( z \in A_i^0 \).

We shall need also the following simple lemma.

1.12. Let \( x, y, z \in L, x \leq y \leq z \), \( y \in A_i(x), y \in A_i^0(z) \). Then there exists \( v \in L \) such that \( v \in A_i^0(x), v \in A_i^0(z) \) and \( v \) is a relative complement of the element \( y \) in the interval \([x, z]\).

Proof. Let \( R_i \) and \( R'_i \) have the same meaning as above. Then \( x \equiv y(R_i) \), \( y \equiv z(R'_i) \). Since \( R_i \) and \( R'_i \) are permutable, there is \( v \equiv v_i \wedge z(R_i) \). Denote \( v = v_i \wedge z \) \( \vee x \). We have \( x \equiv v(R_i) \), \( v \equiv z(R_i) \) and \( x \leq v \leq z \). It remains to prove that \( v \) is a relative complement of \( y \) in \([x, z]\). From \( x \equiv v(v_i) \) we get \( x \equiv y \wedge v(R'_i) \) and analogously from \( x \equiv y(R_i) \) we infer that \( x \equiv y \wedge v(R_i) \), therefore \( x \equiv y \wedge v(R_i) \). Since \( R_i \wedge R'_i \) is the least congruence relation on \( L \), we have \( x = y \wedge v \). By a dual argument, \( z = y \vee v \).

1.13. Remark. The concept of a full subdirect product can be applied in the obvious way for universal algebras. Let \( L \) be a subalgebra of the complete direct product \( \Pi A_i \) of universal algebras \( A_i (i \in I) \) such that the conditions (a) and (b) are fulfilled; then \( L \) is said to be a full subdirect product of algebras \( A_i \). In the case of lattice ordered groups the concept of the weak product (full subdirect product) coincides with the restricted direct product (completely subdirect product [5]) of \( l \)-groups \( A_i \).
2. WEAK PRODUCTS

For the definition of a weak product of universal algebras cf. [2] and the Introduction. We restrict ourselves to the case of lattices; then the definition of this concept is as follows:

Let \( S = \Pi A_i \) be the complete direct product of lattices \( A_i \) and let \( S_1 \) be a sublattice of \( S \) satisfying the following conditions:

(i) if \( x, y \in S_1 \), then the set \( \{ i \in I : x(i) \neq y(i) \} \) is finite;
(ii) if \( x \in S_1 \), \( z \in S \) and if the set \( \{ i \in I : x(i) \neq z(i) \} \) is finite, then \( z \in S_1 \).

Under these assumptions \( S_1 \) is said to be a weak product of lattices \( A_i \). It is easy to verify that any weak product of lattices \( A_i \) is a full subdirect product of these lattices. If the set \( I \) is finite, then the concepts of the weak product, full subdirect product and complete direct product coincide.

2.1. Let \( L \) be a discrete lattice that is a full subdirect product of lattices \( A_i (i \in I) \). Then \( L \) is a weak product of these lattices.

Proof. Let \( x, x_0 \in L \), \( I_1 = \{ i \in I : x(i) = x_0(i) \} \). Further denote \( y = x \wedge x_0 \), \( I_2 = \{ i \in I : x(i) = y(i) \} \). Assume that the set \( I_2 \) is infinite; then there exist distinct elements \( i_1, i_2, i_3, \ldots \in I_2 \). According to (b) for each \( i_k \) there is \( x_k \in L \) such that

\[ x_k(i_k) = x(i_k), \quad x_k(i) = y(i) \quad \text{for each} \quad i \in I, \quad i \neq i_k. \]

Put \( z_n = x_1 \lor x_2 \lor \ldots \lor x_n (n = 1, 2, \ldots) \). Then \( y < z_1 < z_2 < \ldots < x \) and this is impossible, since \( L \) is discrete. Thus \( I_2 \) is finite. Analogously, the set \( I_3 = \{ i \in I : y(i) = z(i) \} \) is finite and therefore the set \( I_4 \) is finite as well and so the condition (i) is fulfilled. Let \( x \in L \), \( z \in \Pi A_i \) \( (i \in I) \) and suppose that the set \( \{ i \in I : x(i) = z(i) \} \) contains only one element \( i_1 \). According to (a) there is \( y \in L \) such that \( y(i_1) = z(i_1) \). Further it follows from (b) that there is \( t \in L \) satisfying \( t(i_1) = y(i_1), \ t(i) = x(i) \) for each \( i \in I, \ i \neq i_1 \). Clearly \( t = z \), thus \( z \in L \). From this we get by induction that (ii) holds.

Remark. Simple examples show that the assertion of the lemma 2.1 need not hold for non-discrete lattices.

If a lattice \( L \) is a full subdirect product of lattices \( A_i \) and \( x_0 \in L \), then we shall write \( L = (fs) \Pi A_i(x_0) \).

Let \( L \) be a lattice. For any \( x_0 \in L \) let \( F(x_0) \) be the system of all sublattices \( A \) of \( L \) such that there exists a full subdirect decomposition

\[
L = (fs) \Pi A_i(x_0) (i \in I)
\]

with the property \( A = A_i(x_0) \) for some \( i_1 \in I \).
2.2. Let $L = (fs) \Pi A_i(x_0) (i \in I), ~ L = (fs) \Pi B_k(x_0) (k \in K), ~ x \in L, ~ i \in I, ~ k \in K, ~ A_i(x_0) = B_k(x_0)$. Then $x(A_i(x_0)) = x(B_k(x_0))$ (i.e., the component of $x$ in any $A \in F(x_0)$ is uniquely determined by $A$).

Proof. Denote $x(A_i(x_0)) = y$, $x(B_k(x_0)) = z$. According to 1.6 $y(B_k(x_0)) = z(A_i(x_0))$. Since $y \in A_i(x_0) = B_k(x_0)$ and analogously $z \in A_i(x_0)$, we have $y(B_k(x_0)) = y$, $z(A_i(x_0)) = z$; thus $y = z$.

Let $L$ be a lattice, $x_0 \in L$. The system of all prime intervals of $L$ will be denoted by $\mathcal{P}$. Let us recall that in §1 we have shown that the following assertion is valid (cf. 1.7 and 1.8):

2.3. If $A, B \in F(x_0)$, then $C = A \cap B \in F(x_0)$ and for any $x \in L$, $x(C) = x(A) \cap x(B)$.

2.4. Let $[u, v] \in \mathcal{P}, A(x_0), C(x_0) \in F(x_0)$ and assume that

$$u(A(x_0)) < v(A(x_0)), \quad u(C(x_0)) < v(C(x_0)).$$

Then $u(A(x_0)) \cap C(x_0)) < v(A(x_0)) \cap C(x_0))$.

Proof. For any $x_1 \in L$ we have

$$u(A(x_0)) < v(A(x_0)) \iff u(A(x_1)) < v(A(x_1)),$$

due to its suffices to prove our statement for the case $x_0 = u$. Under this assumption

$$u(A(x_0)) = u < v(A(x_0)) \leq v$$

and thus (since $[u, v] \in \mathcal{P}$) $v(A(x_0)) = v$. Analogously $v(C(x_0)) = v$. Therefore

$$u(A(x_0)) \cap C(x_0)) = u < v = v(A(x_0)) \cap C(x_0)).$$

If $p = [u, v] \in \mathcal{P}, A \in F(x_0)$ and $u(A) < v(A)$, then $A$ is said to be parallel to $p$. We denote by $F(x_0, p)$ the system of all $A \in F(x_0)$ that are parallel to $p$.

With respect to 2.3 the Lemma 2.4 can be formulated as follows:

2.4'. If $A(x_0), B(x_0) \in F(x_0, p)$, then $A(x_0) \cap B(x_0) \in F(x_0, p)$.

Let us now suppose (in the whole §2) that $L$ is a discrete lattice, card $L > 1$. Then $L$ is conditionally complete. Moreover, $L$ is compact in the following sense: if $u, v \in L$, $u \leq v$ and $\{x_2\} \subseteq [u, v], \forall x \in x_2 \subseteq \{x_2\}$ such that $x_1 \vee \ldots \vee x_n = v$ and dually.

2.5. Let $p \in \mathcal{P}$. For each $x \in L$ there is $A^p(x_0) \in F(x_0, p)$ with the property

(*) if $A_i \in F(x_0, p), A_i \subseteq A^p(x_0), then x(A_i) = x(A^p(x_0))$.

The proof will consist of three steps.
(I) Let \( x_1, x_2 \in L, \ x_1 \leq x_2 \). Let \( R_i = R(A_i) \) and \( R'_i = R'(A_i) \) have the same meaning as in §1. Let \( P \) be the system of all elements \( y \in [x_1, x_2] \) such that

\[ x_i \equiv y(R(A_i)) \quad \text{for each} \quad A_i \in F(x_1, p). \]

Further let \( Q \) be the set of all elements \( z \in [x_1, x_2] \) such that

\[ x_1 \equiv z(R'(A_i)) \quad \text{for some} \quad A_i \in F(x_1, p). \]

Our aim now is to show that the sets \( P \) and \( Q \) have greatest elements (these will by denoted by \( p_0 \) and \( q_0 \), respectively) and that \( p_0 \lor q_0 = x_2 \).

Clearly \( P \) is a convex sublattice of \( L \) and if \( x_1 \leq t \leq z \in Q \), then \( t \in Q \). Let \( z_1, z_2 \in Q \), and let \( x_1, z_2 \in A \), then \( x_1 \equiv z_1(R'(A_{i_0})) \), \( k = 1, 2 \) for some \( A_{i_1}, A_{i_2} \in F(x_1, p) \). In such case according to 2.4' \( A = A_{i_1} \cap A_{i_2} \in F(x_1, p) \), \( R'(A) \geq R'(A_{i_0}) \), \( R'(A) \geq R'(A_{i_1}) \). Hence \( x_1 \equiv z_1 \lor z_2 \). Thus \( z_1 \lor z_2 \in Q \). Therefore \( Q \) is a convex sublattice of \( L \), too. Since \( L \) is conditionally complete and compact, \( P \) has a greatest element \( p_0 \) and analogously \( Q \) possesses a greatest element \( q_0 \). There is \( A_{i_0} \in F(x_1, p) \) such that \( x_1 \equiv q_0(R'(A_{i_0})) \) and clearly \( x_1 \equiv p_0(R'(A_{i_0})) \). Denote \( p_0 \lor q_0 = v \) and assume that \( v < x_2 \). Let \( v_i \in L, \ v < v_i \leq x_2 \) such that \([v, v_i]\) is a prime interval. Suppose, at first, that there exists \( A_i \in F(x_1, p) \) such that \( v \) is not congruent to \( v_i \) mod \( R(A_i) \), then \( v \equiv v_i(R'(A_i)) \). We have \( q_0 \leq v \leq v_i \equiv v(R'(A_i)) \), \( v = v_i(R'(A_i)) \) and thus according to 1.12 there is \( t \in [q_0, v_i] \) such that \( t \land v = q_0, \ t \lor v = v_i, \ q_0 \equiv t \lor v \). Clearly \( q_0 < t \). Put \( A_{i_0} \cap A_i = A \). According to 2.4' \( A \in F(x_1, p) \). Then \( R'(A) \geq R'(A_{i_0}), \ R'(A) \geq R'(A_i) \), therefore \( x_1 \equiv t(R'(A)) \) and so \( t \in Q \); this is a contradiction with the maximality of \( q_0 \) in \( Q \). This shows that we must have \( v \equiv v_i(R'(A_i)) \) for each \( A_i \in F(x_1, p) \). In particular, \( v \equiv v_i(R'(A_{i_0})) \). Clearly \( p_0 \equiv v(R'(A_{i_0})) \). According to 1.12 there is \( t \in L \) such that \( t \land v = p_0, \ t \lor v = v_1 \). The intervals \([p_0, t]\) and \([v, v_i]\) are projective, thus \( p_0 \equiv t(R'(A_i)) \) for each \( A_i \in F(x_1, p) \) and \( p_0 < t \). But then \( x_1 \equiv t(R'(A_{i_0})) \) for each \( A_i \in F(x_1, p) \), whence \( t \in P \) and this is not possible, since \( p_0 \) is the greatest element of \( P \). We have proved that \( p_0 \lor q_0 = x_2 \).

(II) Now let \( x, x_0 \in L, \ x_1 = x \land x_0, \ x_2 = x \lor x_0 \) and denote \( p_1 = x \land p_0, \ q_1 = x \lor q_0. \) From (I) it follows that \( p_1 \) is the greatest element of the interval \([x_1, x]\) with the property that \( x_1 = p_1(R(A_i)) \) for each \( A_i \in F(x_1, p) \) and analogously \( q_1 \) is the greatest element of \([x_1, x]\) with the property \( x_1 = q_1(R(A_i)) \) for some \( A_i \in F(x_1, p) \). Further according to (I) \( x = p_1 \lor q_1 \). Let \( p_2 = x_0 \land p_0, \ q_2 = x_0 \lor q_0. \) For the elements \( x_0, p_2, q_2 \) we can obtain results analogical to those just proved for \( x, p_1, q_1 \); so we have \( x_0 = p_2 \lor q_2 \). Let \( A_{i_0} \) have the same meaning as in (I). Since \( q_1, q_2 \in [x_1, x_0] \), \( p_1, p_2 \in [x_1, x_0] \), the relations \( q_1 \equiv q_2(R'(A_{i_0})), \ p_1 \equiv p_2(R'(A_{i_0})) \) for any \( A_i \in F(x_0, p) \) are valid.

(III) Under the same denotations as in (I) and (II) put \( x^* = p_1 \lor q_2 \). Let \( A_i \in F(x_0, p) \). Then

\[ x = p_1 \lor q_1 \equiv p_1 \lor q_2 = x^*(R'(A_{i_0})). \]
Since $R'(A_{i_0}) \leq R'(A_i)$,

\begin{equation}
(4) \quad x \equiv x^*(R'(A_i)).
\end{equation}

Further we have

\begin{equation}
(4') \quad x_0 = p_2 \lor q_2 \equiv p_1 \lor q_2(R(A_i)).
\end{equation}

From (4) and (4') it follows $x(A_i) = x^* = x(A_{i_0})$ for each $A_i \in F(x_0, p)$, $A_i \subseteq A_{i_0}$. We denote $A_{i_0} = A^*(x_0)$; the proof of the assertion 2.5 is complete.

Now let us denote $A^p = \{x(A^*(x_0)) : x \in L\}$.

2.6. The set $A^p$ is a sublattice of $L$ and the mapping $\varphi_p : x \to x(A^p(x_0))$ is a homomorphism of the lattice $L$ onto $A^p$. For any $x \in A^p$, $\varphi_p(x) = x$.

Proof. Let $x_1, y_1 \in A^p$. There are elements $x, y \in L$ such that $\varphi_p(x) = x_1, \varphi_p(y) = y_1$. Then we have according to 2.4' and 2.5

\[ A = A^p(x_0) \cap A^p(y_0) \cap A^{x \land y}(x_0) \in F(x_0, p), \]

\[ x(A) = x_1, \quad y(A) = y_1, \quad (x \land y)(A) = (x \land y)(A^{x \land y}(x_0)), \]

thus $x_1 \land y_1 = (x \land y)(A^{x \land y}(x_0)) = \varphi_p(x \land y)$. An analogical result holds for $x \lor y$. Hence $A^p$ is a sublattice of $L$ and $\varphi_p$ is a homomorphism of $L$ onto $A^p$. If $x_1 \in A^p, x_1 = x(A^p(x_0))$, then for any $A \subseteq A^p(x_0)$ such that $A \in F(x_0, p)$ we have

\[ x_1(A) = x(A^p(x_0)) \quad (A) = x(A)(A^p(x_0)) = x(A^p(x_0)) \quad (A^p(x_0)) = x(A^p(x_0)) = x_1. \]

For $x_0 \in L, A(x_0) \in F(x_0)$ let $A^*(x_0)$ have the same meaning as in §1.

2.7. Let $x, x_0 \in L, A(x_0) \in F(x_0), B(x_0) \in F(x_0), x(A(x_0)) = x(B(x_0))$. Then $x(A^*(x_0)) = x(B^*(x_0))$.

Proof. Put $x_1 = x \land x_0$ and consider the lattice $[x_1, x]$. It is isomorphic to the direct product $D_1 \times D_2$, where $D_1 = A(x_1) \cap [x_1, x], D_2 = A^*(x_1) \cap [x_1, x]$. The elements $x(A(x_1)), x(A^*(x_1))$ belong to the centre $C_0$ of the lattice $[x_1, x]$ (cf. [1], p. 28) and $x(A(x_1))$ is a complement of $x(A^*(x_1))$; the same holds for $x(B(x_1))$ and $x(B^*(x_1))$. Since $C_0$ is a Boolean algebra and $x(A(x_0)) = x(B(x_0))$ implies $x(A(x_1)) = x(B(x_1))$, we get $x(A^*(x_1)) = x(B^*(x_1))$; from this we obtain $x(A^*(x_0)) = x(B^*(x_0))$.

2.8. For any $x \in L$ and $A_i \in F(x_0, p)$ from $A_i \subseteq A^p(x_0)$ it follows $x(A_i^*) = x(A^*(x_0))$.

This is an immediate consequence of 2.5 and 2.7.

Denote $A^p = \{x(A^p(x_0)) : x \in L\}$. Analogously as in 2.6 we can prove (by using 2.8 instead of 2.5) the proposition:
2.9. The set $A^p*$ is a sublattice of $L$ and the mapping $\varphi_p^* : x \rightarrow x(A^p\ast(x_0))$ is a homomorphism of $L$ onto $A^p$. For any $x \in A^p*$ we have $\varphi_p^*(x) = x$.

For the sake of brevity we denote $x(A^p(x_0)) = x^1$, $x(A^p\ast(x_0)) = x^2$. Let us consider the mapping

$$(x) \rightarrow (x^1, x^2)$$

of the lattice $L$ into $A^p \times A^p*$.

2.10. The mapping $\alpha$ is one-to-one.

Proof. Let $x, y \in L$, $x^1 = y^1$, $x^2 = y^2$ and denote $A^p(x_0) \cap A^p(x_0) = A$. Then $A \in F(x_0, p)$ and according to 2.6 and 2.9 we have

$$x(A) = x(A^p(x_0)) = x^1, \quad x(A^* \ast(x_0)) = x^2$$

and analogously for $y^1, y^2$. Therefore $x(A) = y(A)$, $x(A^*) = y(A^*)$; this implies $x = y$.

2.11. The mapping $\alpha$ is an isomorphism of the lattice $L$ onto $A^p \times A^p*$.

Proof. It suffices to verify that the mapping $\varphi$ is onto. Let $u \in A^p$, $v \in A^p*$. There exist $x, y \in L$ such that

$$u = x(A^p(x_0)), \quad v = y(A^p\ast(x_0)).$$

Put $A = A^p(x_0) \cap A^p(x_0)$. Then $u = x(A)$, $v = y(A^*)$. Thus there is an element $z \in L$ with the property

$$u = z(A), \quad v = z(A^*).$$

Let $B \in F(x_0, p)$, $B \subset A$. Then $z(B) \in A$ and therefore by using 1.7 we obtain

$$z(B) = z(B)(A) = z(A)(B) = u(B) = x(A)(B) = x(A \cap B) = u$$

and according to 2.7 $z(B^*) = v$. Thus $z(A^p(x_0)) = z(A)$, $z(A^p\ast(x_0)) = z(A^*)$. According to the definition of the mapping $\alpha$ this implies $z^1 = u$, $z^2 = v$, $\alpha(z) = (u, v)$.

2.12. $A^p \in F(x_0, p)$.

Proof. Consider the isomorphism $\alpha : L \rightarrow A^p \times A^p*$ and construct $A^p(x_0)$. According to 2.6 $A^p(x_0) = A^p$, thus it suffices to verify that $A^p$ is parallel to the prime interval $p = [c, d]$. But this is equivalent to the assertion that $A^p(c)$ is parallel to $p$ and thus we may assume that $c = x_0$. In such a case $d(A^p) = d > c = c(A^p)$. This shows that $A^p$ is parallel to $p$. 

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2.12.1. \( A^p \subseteq A \) for each \( A \in F(x_0, p) \) and \( x(A^p) = x(A^x(x_0)) \) for each \( x \in L \).

Proof. Let \( y \in A^p, A \in F(x_0, p) \). Then \( y = x(A^x(x_0)) \) for some \( x \in L \). By 2.5, \( A^x(x_0) \) belongs to \( F(x_0, p) \) and thus according to 2.4' \( B = A^x(x_0) \cap A \in F(x_0, p) \), therefore with respect to 2.5 \( x(B) = y \). This implies \( y \in B \subseteq A \), whence \( A^p \subseteq A \). In particular, \( A^p \subseteq A^x(x_0) \) for each \( x \in L \), thus by 2.5 \( x(A^p) = x(A^x(x_0)) \).

2.13. The lattice \( A^p \) is directly indecomposable.

Proof. According to 2.12 card \( A^p \geq 1 \). Let \( x_0 \in L \). Assume (by way of contradiction) at \( A^p \) is directly decomposable. Then there exist lattices \( C_1, C_2 \) with card \( C_1 > 1 \), card \( C_2 > 1 \) such that \( A^p \) is isomorphic to \( C_1 \times C_2 \); thus by 2.11 there is an isomorphism \( f \) of the lattice \( L \) onto \( C_1 \times C_2 \times A^{p*} \) such that \( A^p = C_1(x_0) \times C_2(x_0) \). Either \( C_1(x_0) \) or \( C_2(x_0) \) is paralell to \( p \); we may suppose that \( C_1(x_0) \) satisfies this condition. There exists \( a \in C_2(x_0), a \neq x_0 \). Since \( a \in A^p \), by 2.12.1 we have \( a = a(A^p) = a(A^x(x_0)) \) and therefore for any \( D \in F(x_0, p) \)

\[
D \subseteq A^x(x_0) \Rightarrow a(D) = a.
\]

Put \( D = A^p(x_0) \cap C_1(x_0) \). Clearly \( a(C_1(x_0)) = x_0 \), whence \( a(D) = a(A^x(x_0)) \).

\[ (C_1(x_0)) = a(C_1(x_0)) = x_0 \neq a, \text{ which is a contradiction. The proof is complete.} \]

Let \( \sim \) be an equivalence relation on the set \( P \) defined by

\[ p_1 \sim p_2 \iff A^{p_1} = A^{p_2}. \]

Let \( P_1 \) be a subset of \( P \) containing exactly one element from each equivalence class of the relation \( \sim \). Consider the mapping \( g : L \rightarrow \Pi A^p(p \in P_1) \) defined by the rule

\[ g(x)(p) = x(A^p) \]

(\( x \in L \)). Clearly \( g \) is a homomorphism.

2.14. The mapping \( g \) is one-to-one.

Proof. Assume that there are elements \( x, y \in L \) such that \( x \neq y \), \( g(x) = g(y) = t \); then there is a prime interval \( p_1 = [u, v] \subseteq [x \land y, x \lor y] \) satisfying \( g(u) = g(v) = t \). Hence \( u(A^p) = v(A^p) \) for each \( p \in P_1 \). But there exists \( p_2 \in P \) with \( p_2 \sim p_1 \) and \( u(A^{p_2}) = u(A^{p_1}) \) since \( A^{p_1} \in F(x_0, p_1) \); we have a contradiction.

2.15. The lattice \( L \) is a full subdirect product of lattices \( A^p(p \in P_1) \).

Proof. According to 2.14 the mapping \( g \) is an isomorphism of \( L \) into \( \Pi A^p(p \in P_1) \). Since each \( A^p \) is a direct factor of \( L, x_0 \in A^p \), the conditions (a) and (b) from §1 are fulfilled.
From 2.15 and 2.1 it follows:

2.16. Theorem. Any discrete lattice is a weak product of directly indecomposable lattices.

Since any two full subdirect decompositions have a common refinement, the representation of a discrete lattice as a weak product of indecomposable lattices is unique.

3. ISOMORPHISMS OF UNORIENTED GRAPHS OF DISCRETE LATTICES

Let $L$ be a discrete lattice and let $\mathcal{P}$ be the set of all prime intervals of $L$. We denote by $G(L)$ the unoriented graph such that the set of vertices of $G(L)$ equals $L$ and two vertices $x, y$ of $G(L)$ are assumed to be joined by an edge if and only if either $[x, y] \in \mathcal{P}$ or $[y, x] \in \mathcal{P}$. In [1] there is formulated the following problem (Problem 8): what discrete lattices $L$ satisfy the condition that for each discrete lattice $L'$ the implication

$$G(L) \sim G(L') \Rightarrow L \sim L'$$

is valid (where $\sim$ denotes the isomorphism of graphs or lattices, respectively). The answer to this problem for general lattices is unknown. In [4] a solution for the case of finite modular lattices was given. Now we shall prove that the result of [4] can be generalized for infinite modular lattices.

For any lattice $L$ we denote by $\bar{L}$ the lattice that is dual to $L$. $L$ is self-dual, if $L \sim \bar{L}$.

The following propositions 3.1 and 3.2 are known [4]:

3.1. Let $L$ and $L'$ be discrete modular lattices. Then the following conditions are equivalent:

(i) $G(L) \sim G(L')$.

(ii) There are lattices $A, B$ such that $L \sim A \times B, L' \sim A \times \bar{B}$.

3.2. Let $L$ be a finite modular lattice. Then the following conditions are equivalent:

(i) For any finite modular lattice $L'$ the implication (5) is valid.

(ii) If $L \sim A \times B$, then $A \sim \bar{A}$ (i.e., each direct factor of $L$ is self-dual).

Our aim now is to show that the assertion of 3.2 remains valid for infinite discrete modular lattices.

3.3. Let $L$ be a discrete lattice, $x_0 \in L$, and let $L$ be a full subdirect product of lattices $A_i (i \in I)$. Let $I = I_1 \cup I_2, I_1 \cap I_2 = \emptyset$,

$$A = \{ x \in L : i \in I_2 \Rightarrow x(A_i(x_0)) = x_0 \},$$

$$B = \{ x \in L : i \in I_1 \Rightarrow x(A_i(x_0)) = x_0 \}.$$

Then $L \sim A \times B$. 410
Proof. Let \( x \in L \) and for \( i \in I \) let us write \( x(i) \) rather than \( x(A_i(x_0)) \). According to 2.1 the set \( \{ i \in I : x(i) = x_0 \} \) is finite and so it follows from §1, (b) and by induction that there is an element \( x_1 \in A \) with the property \( x(i) = x_1(i) \) for each \( i \in I_1 \). Analogously there exists \( x_2 \in B \) satisfying \( x(i) = x_2(i) \) for each \( i \in I_2 \). The mapping \( \psi : x \mapsto (x_1, x_2) \) is obviously an isomorphism of the lattice \( L \) into \( A \times B \). Let \( a \in A \), \( b \in B \). Since the sets \( \{ i \in I : a(i) = x_0 \} \), \( \{ i \in I : b(i) = x_0 \} \) are finite and disjoint, with respect to (b) we can find \( x \in L \) such that \( x_1 = a \) and \( x_2 = b \); this shows that \( \psi \) is onto.

The assertion of the proposition 3.3 need not hold for the case when \( L \) is not discrete.

3.4. Let \( L \) be a discrete modular lattice such that each directly indecomposable direct factor of \( L \) is self-dual. Then for each discrete modular lattice \( L' \) the implication (5) holds.

Proof. Let \( L' \) be a discrete modular lattice, \( G(L) \sim G(L') \). According to 3.1 there exist lattices \( A, B \) such that \( L \sim A \times B \), \( L' \sim \overline{A} \times B \). Then \( A \) is isomorphic to a sublattice of \( L \), whence \( A \) is discrete. By 2.16 \( A \) is a full subdirect product of directly indecomposable lattices \( A_i \). Any \( A_i \) is isomorphic to a directly indecomposable direct factor of \( L \) and therefore \( A_i \sim \overline{A_i} \). From this it follows \( A \sim \overline{A} \) and hence \( L \sim L' \).

3.5. Let \( L \) be a discrete modular lattice such that for any discrete modular lattice \( L' \) the implication (5) is valid. Then each direct factor of \( L \) is self-dual.

Proof. Since any direct factor \( A \) of \( L \) is discrete and thus \( A \) is a full subdirect product of directly indecomposable factors it suffices to prove that each directly indecomposable direct factor of \( L \) is self-dual. Let \( A_0 \) be a directly indecomposable direct factor of \( L \) and assume (by way of contradiction) that \( A_0 \) is not self-dual. Let \( x_0 \in L \). We may assume that \( A_0 \in F(x_0) \). There exists a full subdirect decomposition

\[
L = (fs) \Pi A_i(x_0) (i \in I),
\]

where all factors \( A_i(x_0) \) are directly indecomposable. Put

\[
I_1 = \{ i \in I : A_i(x_0) \sim A_0 \}, \quad I_2 = I \setminus I_1
\]

and let \( A, B \) have the same meaning as in 3.3. According to 3.3 \( L \sim A \times B \), whence by 3.1 \( G(L) \sim G(L') \), where \( L = \overline{A} \times B \). Thus if \( X \) is a directly indecomposable direct factor of \( L' \), then \( X \) is isomorphic to some \( A_i(x_0) (i \in I_2) \) or to some \( \overline{A_i(x_0)} (i \in I_1) \); therefore \( X \) cannot be isomorphic to \( A_0 \). From this it follows that \( L' \) is not isomorphic to \( L \), which is a contradiction.

3.6. Let \( L \) be a full subdirect product of directly indecomposable lattices \( A_i(i \in I) \). If all lattices \( A_i \) are self-dual, then all direct factors of \( L \) are self-dual.
Proof. Let $x \in L$. Let $L$ be isomorphic to a direct product $A \times B$. Then because any two full subdirect decompositions of $L$ have a common refinement (Theorem 1.11) and since $A_i(x_0)$ are directly indecomposable, there is a subset $I_1 \subseteq I$ such that $A$ is a full subdirect product of lattices $A_i(x_0)$ ($i \in I_1$). Because $A_i(x_0)$ are self-dual, so is the lattice $A$.

By summarizing, we get from 3.4, 3.5 and 3.6:

3.7. Theorem. Let $L$ be a discrete modular lattice. Then the following conditions are equivalent:

(i) For any discrete modular lattice $L'$ the implication (5) is fulfilled.
(ii) Each directly indecomposable direct factor of $L$ is self-dual.
(iii) Each direct factor of $L$ is self-dual.

Let $L$ and $L'$ be finite lattices such that $G(L) \sim G(L')$. If $G$ is modular, then so is $L$ [4]. It remains as an open question whether this assertion is valid for infinite discrete lattices.

References


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