Ladislav Bican
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Czechoslovak Mathematical Journal, Vol. 21 (1971), No. 3, 413–417

Persistent URL: `http://dml.cz/dmlcz/101041`

Czechoslovak Mathematical Journal, 21 (96) 1971, Praha

A NOTE ON MIXED ABELIAN GROUPS

LADISLAV BICAN, Praha

(Received April 2, 1970)

The purpose of this note is to give a criterion for splittingness of some class of mixed abelian groups. Similar problems are studied in the papers [4] and [5]. This work continues on the ideas of my paper [2].

By the word "group" we shall always mean an additively written abelian group. As in [2] we use the notions "characteristic" and "type" in the large sense, i.e. we deal with this notions in mixed groups. However, it is clear that some its properties do not hold in general. The symbol $h_p^G(g)$ ($\tau^G(g)$, $\hat{\tau}^G(g)$ resp.) denotes the p-height (the characteristic, the type resp.) of the element g in the group G. If T is a torsion group, then T_p will denote the p-primary component of T and similarly if Π is a set of primes, then T_Π is defined by $T_\Pi = \sum_{p \in \Pi}{}_d\, T_p$. In the next we shall deal with mixed group G with maximal torsion subgroup T and $\bar{G}$ will denote the factor-group G/T. The bar over the elements will denote the elements from $\bar{G}$. For the sake of simplicity we shall write briefly $\tau(g)$, $\tau(\bar{g})$, $\hat{\tau}(g)$ etc. in place of $\tau(g)$, $\tau^{\bar{G}}(\bar{g})$, $\hat{\tau}^G(g)$ etc.

In general, we shall adopt the notation used in [1].

At first, we shall formulate the following conditions:

Condition (α): We say, that a mixed group G with maximal torsion subgroup T satisfies the condition (α) if to any $g \in G \dot{-} T$ there exists an integer m (depending on g, of course), such that $\hat{\tau}(mg) = \hat{\tau}(\bar{g})$.

Condition (β): We say, that a mixed group G with maximal torsion subgroup T satisfies the condition (β) if to any $g \in G \dot{-} T$ there exists an integer m, such that for any prime p with $h_p(\bar{g}) = \infty$ there exist the elements $h_0^{(p)} = mg$, $h_1^{(p)}$, $h_2^{(p)}$, ..., such that $ph_{n+1}^{(p)} = h_n^{(p)}$, $n = 0, 1, 2, \ldots$

Condition (γ): We say, that a mixed group G satisfies the condition (γ) if it holds: if $\bar{G} = G/T$ contains an element of infinite p-height, then T_p is a direct sum of a divisible and a bounded groups.

Condition (δ): We say, that a mixed group G with maximal torsion subgroup T satisfies the condition (δ) if for any prime p there is $T_p = 0$ whenever $\bar{G}$ is not p-divisible.

Remark. For the conditions (α), (β), (γ) see [2] and for the condition (δ) see [5].

Now we shall formulate two preliminary lemmas:

Lemma 1. *Let G be a mixed group with maximal torsion subgroup T and $\bar{0} \neq \bar{h} \in \bar{G}$ be an arbitrary element. If for some $x \in \bar{h}$ there is $\tau(x) = \tau(\bar{h})$, then $\tau(\varrho x) = \tau(\varrho \bar{h})$ for any integer ϱ. Similarly, if for some $x \in \bar{h}$ there is $\hat{\tau}(x) = \hat{\tau}(\bar{h})$ then $\hat{\tau}(\varrho x) = \hat{\tau}(\varrho \bar{h})$ for any integer ϱ.*

Proof. See Lemma 1 in [2].

Lemma 2. *Let G be a mixed group with maximal torsion subgroup T. If G satisfies the condition (α), then to any $g \in G \dot{-} T$ there exists a non-zero integer ϱ and an element $h \in \bar{g}$ such that $\tau(\varrho h) = \tau(\varrho \bar{g})$.*

Proof. See Lemma 3 in [2].

Now we are ready to prove the main result:

Theorem 1: *Let G be a mixed group with maximal torsion subgroup T satisfying the condition (δ). Then the conditions (α) and (β) are necessary and sufficient for G to be split.*

Proof. The conditions (α) and (β) are necessary by Lemma 4 from [2].

The proof of sufficiency we shall make in several steps:

a) Let $\{\bar{h}_\lambda; \lambda \in \Lambda\}$ be a basis of $\bar{G}$. In view of Lemmas 1 and 2 we can assume that any $\bar{h}_\lambda$ contains an element h_λ such that $\tau(h_\lambda) = \tau(\bar{h}_\lambda)$, $\lambda \in \Lambda$ and that the integers m_λ belonging to h_λ under the condition (β) are all equal to 1. For the sake of simplicity let us denote $h_p(h_\lambda) = n_\lambda(p)$ for all primes p and all $\lambda \in \Lambda$ such that $n_\lambda(p)$ is either a natural integer or zero or the symbol ∞. For all primes p and all $\lambda \in \Lambda$ let us define the elements $h^{(p)}_{\lambda,k}$, $0 \leqq k < n_\lambda(p) + 1$ (where $\infty + 1 = \infty$) in the following way: $h^{(p)}_{\lambda,0} = h_\lambda$ for all primes p. If $n_\lambda(p)$ is a natural integer, then $h^{(p)}_{\lambda,n_\lambda(p)}$ be some solution of the equation $p^{n_\lambda(p)}x = h_\lambda$ lying in G and for $1 \leqq k < n_\lambda(p)$ we put $h^{(p)}_{\lambda,k} = p^{n_\lambda(p)-k}h^{(p)}_{\lambda,n_\lambda(p)}$. If $n_\lambda(p) = \infty$, then $h^{(p)}_{\lambda,k}$ be such elements of G the existence of which (for h_λ) is guaranted by the condition (β). Now we shall define the subgroup H of G as follows: $H = \{h^{(p)}_{\lambda,k}; \lambda \in \Lambda, p$ runs over all primes and $0 \leqq k < n_\lambda(p) + 1\}$. Let p be an arbitrary prime for which $\bar{G}$ is not p-divisible. Then any equation of the form

$$p^k x = h\,, \quad h \in H \tag{1}$$

has in G at most one solution. Clearly, if x, y are solutions of (1), then $p^k(x - y) = 0$, hence $x - y \in T_p = 0$.

Finally, we shall define the subgroup A of G as the subgroup generated by H and all solutions of the equations (1) (if they exist) for all natural integers k, $h \in H$ and all primes p for which $\bar{G}$ is not p-divisible.

b) It is not too hard to prove that A is just the set of all elements a of G for which there exists a natural integer m relatively prime to any prime p for which $\bar{G}$ is p-divisible such that $ma \in H$. (The proof is analogical as the part a) of the proof of Lemma 12 in [2].)

c) We shall prove now that

$$H \cap T = 0\,. \tag{2}$$

Let us note that if $h^{(p)}_{\lambda,i_1}; h^{(p)}_{\lambda,i_2}; \ldots; h^{(p)}_{\lambda,i_k}$; $i_1 < i_2 < \ldots < i_k$ are generators of H belonging to some prime p and $\lambda \in \Lambda$, then clearly $\sum_{j=1}^{k} \mu_j h^{(p)}_{\lambda,i_j} = (\sum_{j=1}^{k} \mu_j p^{i_k - i_j})\, h^{(p)}_{\lambda,i_k}$.

If g is an arbitrary element of $H \cap T$ then g can be written in the form: $g = \sum_{i=1}^{n} \sum_{j=1}^{l_i} \mu_{ij} h^{(p_{i,j})}_{\lambda_i,k_{i,j}}$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are different elements from Λ and for any $i = 1, 2, \ldots, n$, $p_{i,1}; p_{i,2}; \ldots; p_{i,l_i}$ are different primes. If we put $r_i = p_{i,1}^{k_{i,1}} \cdot p_{i,2}^{k_{i,2}} \cdot \ldots \cdot p_{i,l_i}^{k_{i,l_i}}$, $i = 1, 2, \ldots, n$ and $r = r_1 \cdot r_2 \cdot \ldots \cdot r_n$, we can define the integers $\bar{r}_i$ by the formula $r = r_i \cdot \bar{r}_i$. Then it holds $rg = \sum_{i=1}^{n} \bar{r}_i \sum_{j=1}^{l_i} \mu_{ij} r_i h^{(p_{i,j})}_{\lambda_i,k_{i,j}} = \sum_{i=1}^{n} (\bar{r}_i \sum_{j=1}^{l_i} \mu_{ij}(r_i/p_{i,j}^{k_{i,j}}))\,.\, h_{\lambda_i} \in H \cap T$. In view of the independence of $\bar{h}_\lambda$ in $\bar{G}$ it is $\bar{r}_i \sum_{j=1}^{l_i} \mu_{ij}(r_i/p_{i,j}^{k_{i,j}}) = 0$, $i = 1, 2, \ldots, n$ from which (due to $\bar{r}_i \neq 0$) it follows $\mu_{ij} = p_{i,j}^{k_{i,j}} \mu'_{ij}$ and hence $g = \sum_{i=1}^{n} (\sum_{j=1}^{l_i} \mu'_{ij})\, h_{\lambda_i} \in H \cap T$. From the independence of $\bar{h}_\lambda$ in $\bar{G}$ it follows $\sum_{j=1}^{l_i} \mu'_{ij} = 0$, $i = 1, 2, \ldots, n$, so that $g = 0$.

d) We proceed to show that

$$A \cap T = 0\,. \tag{3}$$

Let $g \in A \cap T$ be an arbitrary element. By the part b) there is $mg \in H \cap T$ for a suitable positive integer m relatively prime to any prime p for which $\bar{G}$ is p-divisible. By (2) it is $mg = 0$. Let $m = p_1^{k_1} \cdot p_2^{k_2} \cdot \ldots \cdot p_n^{k_n}$ be the canonical decomposition of m. It is easy to see that there exist integers η_i, $i = 1, 2, \ldots, n$ with $\sum_{i=1}^{n} (m/p_i^{k_i})\, \eta_i = 1$ Then $g = \sum_{i=1}^{n} (m/p_i^{k_i})\, \eta_i g$. At the same time there is $p_i^{k_i}(m/p_i^{k_i})\, \eta_i g) = 0$, i.e. $(m/p_i^{k_i})\, \eta_i g \in T_{p_i}$, $i = 1, 2, \ldots, n$. By the condition (δ) there is $T_{p_i} = 0$, $i = 1, 2, \ldots, n$, so that $g = 0$.

e) Our Theorem will be proved if we show that

$$G = \{A, T\}\,. \tag{4}$$

Let $g \in G$ be an arbitrary element. The elements $\bar{h}_\lambda$, $\lambda \in \Lambda$ forms a basis of $\bar{G}$, such that there exist the integers (not all equal zero) $\varrho, \mu_1, \mu_2, \ldots, \mu_n$ for which $\varrho\bar{q} = \sum_{i=1}^{n} \mu_i \bar{h}_{\lambda_i}$. Then

$$\varrho g = \sum_{i=1}^{n} \mu_i h_{\lambda_i} + t = h + t\,, \quad t \in T\,. \tag{5}$$

The integer ϱ can be written in the form $\varrho = \varrho_1 \cdot \varrho_2$ where ϱ_1 is divided just by those primes, for which $\bar{G}$ is p-divisible and hence ϱ_2 is divided just by those primes for which $\bar{G}$ is not p-divisible. By the definition of H there exists an element $h' \in H$ for which

$$h = \varrho_1 h'\,. \tag{6}$$

By the condition (δ) the group T is ϱ_2-divisible, such that there exists an element $t' \in T$ for which

$$t = \varrho_2 t'\,. \tag{7}$$

From (5), (6), (7) it follows $\varrho g = \varrho_1 h' + \varrho_2 t'$, hence $\varrho_2(\varrho_1 g - t') = \varrho_1 h'$ so that by b) $\varrho_1 g - t' \in A$. We can write now

$$\varrho_1 g = a + t'\,, \quad a \in A\,, \quad t \in T\,. \tag{8}$$

From the parts a) and b) of this proof it easily follows that A is ϱ_1-divisible. Hence there exists an element $a' \in A$ for which $a = \varrho_1 a$. From this and from (8) $\varrho_1(g - a') = t' \in T$ hence $g - a' \in T$ and the proof is finished.

Theorem 2. *Let G be a mixed group with maximal torsion subgroup T and let us denote by Π the set of all primes p for which $\bar{G}$ is not p-divisible. Then G splits if and only if*

1) *T_Π is direct summand of G,*
2) *G satisfies the conditions (α) and (β).*

Proof. The necessity of these conditions is obvious.

Proceeding to the sufficiency let us assume that T_Π is a direct summand of G and that the conditions (α) and (β) are satisfied. Thus we have

$$G = T_\Pi \dotplus G_1 \tag{9}$$

where G_1 is mixed. Its maximal torsion subgroup we denote by Q and G_1/Q we denote by $\bar{G}_1$. At first, it is easy to show that the correspondence $g + Q \leftrightarrow g + T$ for $g \in G_1$ is an isomorphism between G_1/Q and G/T, which together with the condition (α) for G implies that $\hat{\tau}(mg) = \hat{\tau}(\bar{g}) = \hat{\tau}^{\bar{G}_1}(\bar{g}) \geqq \hat{\tau}^{G_1}(mg) = \hat{\tau}(mg)$ for any $g \in G_1 \dot{-} Q$ and a suitable integer m (depending on g), such that the condition (α) is satisfied for G_1.

At second, let $g \in G_1 \dotdiv Q$ be an arbitrary element. By the condition (β) (for G) there exist an integer m such that for any prime p with $h_p(\bar{g}) = \infty$ there exist the elements $h_0^{(p)} = mg, h_1^{(p)}, h_2^{(p)}, \ldots$ such that $ph_{n+1}^{(p)} = h_n^{(p)}$, $n = 0, 1, 2, \ldots$ By (9) we can write $h_i^{(p)} = t_i^{(p)} + g_i^{(p)}$, $t_i^{(p)} \in T_\Pi$, $g_i^{(p)} \in G_1$, $i = 0, 1, 2, \ldots$ It holds $ph_{n+1}^{(p)} = pt_{n+1}^{(p)} + pg_{n+1}^{(p)} = h_n^{(p)} = t_n^{(p)} + g_n^{(p)}$, hence $pg_{n+1}^{(p)} = g_n^{(p)}$ and the condition (β) is satisfied for G_1.

Finally, in view of the isomorphism $G_1/Q \cong G/T$ it holds: if p is an arbitrary prime for which G_1/Q is not p-divisible, then $\bar{G}$ is not p-divisible, too, and $p \in \Pi$. Hence $Q_p = 0$ and G_1 satisfies the condition (δ). Now it suffices to use Theorem 1.

Corollary 1. *Let G be a mixed group with maximal torsion subgroup T. Let us denote by Π the set of all primes p for which $\bar{G}$ is not p-divisible. If T_Π si a direct sum of a divisible and a bounded groups and the conditions (α) and (β) are satisfied for G, then G splits.*

Proof. Clearly, by Theorems 18.1 and 24.5 from [1] T_Π is a direct summand of G.

Corollary 2. *Let G be a mixed group with maximal torsion subgroup T. If G/T is divisible and G satisfies the conditions (α) and (β), then G splits.*

Theorem 3. *Let G be a mixed group with maximal torsion subgroup T satisfying the condition (δ). Then any pure subgroup of G splits if and only if the conditions (α) and (γ) are satisfied.*

Proof. Let $\bar{g} \in \bar{G}$ be an element of infinite p-height (p is a prime). Then the inverse image S of the pure subgroup of $\bar{G}$ generated by $\bar{g}$ under the natural homorphism of G onto $\bar{G}$ is of torsion free rank one pure in G, and the necessity of the condition (γ) now easily follows from Theorem 4 from [2] (and from the transitivity of purity), while the necessity of the condition (α) is obvious.

Conversely, if S is a pure subgroup of G, then by Lemma 6 from [2] S satisfies the conditions (α) and (δ). By Lemma 10 from [2] S satisfies the condition (γ) and hence by Lemma 5 from [2] S satisfies the condition (β). Theorem 1 now finishes the proof.

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Author's address: Praha 8 - Karlín, Sokolovská 83, ČSSR (Matematicko-fyzikální fakulta KU).