Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 21 (1971), No. 3, 418–423

Persistent URL: http://dml.cz/dmlcz/101042

Czechoslovak Mathematical Journal, 21 (96) 1971, Praha

ON APPROXIMATION OF BAIRE FUNCTIONS BY DARBOUX FUNCTIONS

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(Received April 15, 1970)

1. TERMINOLOGY AND NOTATION

Throughout this paper, unless otherwise specified, all functions will supposed to be real valued defined on a (possibly infinite) interval I.

We use the usual Borel classification of sets (see Kuratowski [4], page 250).

The class of Borel α functions is denoted as $\mathscr{B}_\alpha$ while the class of Baire α functions as Φ_α. As is well-known, for α finite we have $\Phi_\alpha = \mathscr{B}_\alpha$, but $\Phi_\alpha = \mathscr{B}_{\alpha+1}$ for α infinite. (For facts concerning Borel and Baire functions see KURATOWSKI [4], page 280, resp. 306.) $\mathscr{D}$ stands for the class of Darboux functions. For two classes $\mathscr{A}$ and $\mathscr{B}$ of functions let $\mathscr{A}\mathscr{B}$ denote the class $\mathscr{A} \cap \mathscr{B}$, e.g. $\mathscr{D}\mathscr{B}_\alpha$.

All limits of sequences of functions are pointwise limits. If $\mathscr{A}$ is a class of functions then $\mathscr{A}\uparrow$ (resp. $\mathscr{A}\downarrow$) denotes the set of all functions which are limits of increasing (resp. decreasing) sequences of functions in $\mathscr{A}$. Finally we write $\mathscr{A}\uparrow\downarrow$ for $(\mathscr{A}\uparrow)\downarrow$ and similarly with $\mathscr{A}\downarrow\uparrow$.

2. INTRODUCTION

It is known that each $f \in \Phi_\alpha$ with $\alpha \geqq 1$ is the limit of a sequence $\{f_n\}_{n=1}^{\infty}$ of functions such that each f_n is in $\mathscr{D}\Phi_{\alpha-1}$ if α is a non-limit ordinal, and $f_n \in \bigcup_{\beta<\alpha} \mathscr{D}\Phi_\beta$ otherwise (see [3], [5], [1], [6], and [7]). In the present paper a somewhat sharper result is given: For each ordinal $\alpha \geqq 1$, there is a lattice Ω_α of Darboux functions in Baire classes preceeding α such that Φ_α is the pointwise closure of Ω_α (see Theorem 2 below; the case $\alpha = 2$ is a simple consequence of Preiss' result [7]).

The following theorems have been stated in [2] by Ceder and Weiss:

Theorem A. $\mathscr{D}\uparrow\downarrow = \mathscr{D}\downarrow\uparrow$ *is the class of all functions.*

Theorem B. *Let $\alpha \geqq 1$. Then*

$$\Phi_\alpha = (\Phi_{\alpha-1}\uparrow\downarrow) \cap (\Phi_{\alpha-1}\downarrow\uparrow)$$

for α a non-limit ordinal and

$$\Phi_\alpha \supset \Big(\bigcup_{\beta<\alpha} (\Phi_\beta\uparrow)\Big)\downarrow \cap \Big(\bigcup_{\beta<\alpha} (\Phi_\beta\downarrow)\Big)\uparrow$$

when α is a limit ordinal.

The autors claimed that $\Phi_\alpha = \big(\bigcup_{\beta<\alpha} (\Phi_\beta\uparrow)\big)\downarrow \cap \big(\bigcup_{\beta<\alpha} (\Phi_\beta\downarrow)\big)\uparrow$ when α is a limit ordinal. Their proof is invalid and the question of equality remains open.

In connection with these results the following problem is posed in [2] by Ceder and Weiss: What is the class $\mathcal{D}\mathcal{B}_1\downarrow\uparrow \cap \mathcal{D}\mathcal{B}_1\downarrow\uparrow$? From a result [7] of D. Preiss it follows that $\mathcal{D}\mathcal{B}_1\uparrow\downarrow \cap \mathcal{D}\mathcal{B}_1\downarrow\uparrow = \mathcal{B}_2$. In the present paper it is shown that, in harmony with the above cited Theorem B a similar result holds for each class Φ_α where α is a non-limit ordinal > 0 (see Theorem 3 below).

In the above cited paper [2] Ceder and Weiss give a characterization of the classes $\mathcal{D}\uparrow$ and $\mathcal{D}\downarrow$. In the present paper a similar characterization of the classes $(\mathcal{D}\Phi_\alpha)\uparrow$ and $(\mathcal{D}\Phi_\alpha)\downarrow$ with $\alpha > 1$ is given (see Theorem 1 below).

3. APPROXIMATION THEOREMS

We begin with two lemmas.

Lemma 1. *Let $\{(I_n, A_n)\}_{n=1}^\infty$ be a sequence of ordered pairs such that each I_n is an open interval and A_n a Borel set, and let for each n, the set $I_n \cap A_n$ be uncountable. Then there is a non-empty nowhere dense perfect set $P \subset I_1 \cap A_1$ such that for each n, the set $I_n \cap A_n - P$ is uncountable.*

Proof. Since $I_1 \cap A_1$ is an uncountable Borel set, it contains a non-empty nowhere dense perfect subset B (see Kuratowski [4], page 387). Define for each $n > 1$, the set C_n in this way: If the set $B \cap I_n \cap A_n$ is at most countable, let $C_n = \emptyset$. Otherwise $B \cap I_n \cap A_n$ as uncountable Borel set contains a non-empty perfect subset and hence a non-empty perfect subset which is nowhere dense in B. Denote this non-empty perfect set by C_n. Since $\bigcup_{n=1}^\infty C_n$ is of the first category in B and B is closed, the set $B - \bigcup_{n=1}^\infty C_n$ is non-empty. Moreover, it is an uncountable (B has no isolated points) Borel set. Hence there exists a non-empty nowhere dense perfect subset, say P, contained in $B - \bigcup_{n=1}^\infty C_n$. It is easy to verify that for each n, the set $I_n \cap A_n - P$ is uncountable, q.e.d.

Lemma 2. *Let $\{(I_n, A_n)\}_{n=1}^{\infty}$ be a sequence of ordered pairs such that I_n is an open interval and A_n a Borel set and assume for each n, that the set $I_n \cap A_n$ is uncountable. Then there are disjoint non-empty nowhere dense perfect sets $\{P_n\}_{n=1}^{\infty}$ such that $P_n \subset I_n \cap A_n$, for each n.*

Proof. By Lemma 1, there exists a non-empty nowhere dense perfect set P_1 such that $P_1 \subset I_1 \cap A_1$ and each set $I_n \cap A_n - P_1 = I_n \cap (A_n - P_1)$ is uncountable. In general, by induction let $P_1, P_2, \ldots, P_k$ be disjoint non-empty nowhere dense perfect sets such that $P_i \subset I_i \cap A_i$, $i = 1, 2, \ldots, k$, and for each n, $I_n \cap (A_n - \bigcup_{i=1}^{k} P_i)$ is uncountable. By applying the Lemma 1 to the sets $\{(I_n, A_n - \bigcup_{i=1}^{k} P_i)\}_{n=k+1}^{\infty}$ obtain a non-empty nowhere dense perfect set P_{k+1}. It is easy to verify that $P_{k+1} \subset I_{k+1} \cap \cap A_{k+1}$, that for each n, $I_n \cap (A_n - \bigcup_{i=1}^{k+1} P_i)$ is uncountable and that $P_1, P_2, \ldots, P_{k+1}$ are disjoint.

Now we are able to prove the following

Theorem 1. *For each ordinal $\alpha > 1$,*

$$(\mathscr{D}\Phi_\alpha)\uparrow = (\mathscr{D}\uparrow) \cap (\Phi_\alpha\uparrow) \quad \text{and} \quad (\mathscr{D}\Phi_\alpha)\downarrow = (\mathscr{D}\downarrow) \cap (\Phi_\alpha\downarrow).$$

Proof. To prove the theorem it suffices to show that $(\mathscr{D}\uparrow) \cap (\Phi_\alpha\uparrow) \subset (\mathscr{D}\Phi_\alpha)\uparrow$. (The proof for $(\mathscr{D}\downarrow) \cap (\Phi_\alpha\downarrow) \subset (\mathscr{D}\Phi_\alpha\downarrow)$ is similar.) We can without loss of generality assume that all functions in the sequel are defined on an open interval I. Let $f \in \in (\mathscr{D}\uparrow) \cap (\Phi_\alpha\uparrow)$. Let $\{(I_n, J_n)\}_{n=1}^{\infty}$ be an enumeration of all pairs (I_n, J_n) of intervals I_n, J_n with rational end-points, where I_n are open intervals which are contained in I, and J_n are intervals of the form $(r, r'\rangle$ $(= \{x; r < x \leqq r'\})$, and such that $I_n \cap \cap f^{-1}(J_n)$ is uncountable. Apply the Lemma 2 to obtain a sequence $\{P_n\}_{n=1}^{\infty}$ of disjoint non-empty nowhere dense perfect sets such that for each n, $P_n \subset I_n \cap f^{-1}(J_n)$.

If r_n is the left-side end-point of the interval J_n, let g_n be a continuous function defined on P_n which maps P_n onto the closed interval $\langle \min(-n, r_n - n), r_n \rangle$. Since f is in $\Phi_\alpha\uparrow$, there exists an increasing sequence $\{f'_n\}_{n=1}^{\infty}$ of Baire α functions such that $f = \lim_{n\to\infty} f'_n$. Define functions $\{f''_n\}_{n=1}^{\infty}$ as follows:

$$f''_n(x) = \begin{cases} g_m(x) & \text{if} \quad x \in P_m \quad \text{and} \quad m \geqq n\,, \\ f'_n(x) & \text{if} \quad x \notin \bigcup_{m=n}^{\infty} P_m\,. \end{cases}$$

Finally, for each n, let

$$f_n(x) = \max\,(f''_1(x), f''_2(x), \ldots, f''_n(x))\,.$$

It is easy to see that $\{f_n\}_{n=1}^{\infty}$ is an increasing sequence of functions such that $\lim_{n\to\infty} f_n = f$.

As is well-known, the set Φ_α is the set of all Borel α functions if α is finite, and Φ_α is the set of all Borel $\alpha + 1$ functions if α is infinite (see [4], page 299). Hence to show that $f_n \in \Phi_\alpha$ it suffices to show that for each real λ, $[f''_n < \lambda]$ and $[f''_n > \lambda]$ are of the additive Borel class β where $\beta = \alpha$ if α is finite and $\beta = \alpha + 1$ otherwise. The fact that $[f''_n < \lambda]$ is of the additive Borel class β follows from the equality

$$[f''_n < \lambda] = ([f'_n < \lambda] \cap (I - \bigcup_{i=n}^{\infty} P_i)) \cup (\bigcup_{i=n}^{\infty} [g_i < \lambda]),$$

and from the fact that the first set on the right-hand side of this equality is of the additive Borel class β while the second set is of the type F_σ. The argument is similar for $[f''_n > \lambda]$.

Finally, each f_n is in $\mathscr{D}$. To see it assume that $x < y$ and (say) $f_n(x) < \xi < f_n(y)$, where ξ is a real number (in the case $f_n(x) > f_n(y)$ the proof is similar). Since $\xi < f_n(y) \leqq f(y)$ we have $y \in [f > \xi]$. Since f is in $\mathscr{D}\!\uparrow$ the set $[f > \xi]$ is bilaterally c-dense in itself (see [2], Corollary 2 of Th. 3). Hence

$$\text{card}\,([f > \xi] \cap (x, y)) = c \tag{1}$$

(here c denotes the cardinality of the continuum). Let l be a natural number such that $-l < \xi$. From (1) it follows that there is a member (P_q, J_q) in the sequence $\{(I_k, J_k)\}_{k=l+n}^{\infty}$ such that $I_q \subset (x, y)$ and $J_q \subset (\xi, +\infty)$. Now from the definition of g_q we have $g_q(z) = \xi$ for some $z \in P_q \subset I_q$ and hence for some $z \in (x, y)$. Since $q > n$ it follows from the definition of the function f_n that $f_n(z) = g_q(z) = \xi$. Thus we have shown that f is the limit of an increasing sequence of functions in $\mathscr{D}\Phi_\alpha$, q.e.d.

The next Theorem 2 is an extension of results found in [3], [5], [1], [6], and [7].

Theorem 2. *For each ordinal $\alpha > 0$ there is a lattice Ω_α of functions defined on an interval I such that $\Omega_\alpha \subset \bigcup_{\beta<\alpha} \mathscr{D}\Phi_\beta$, and Φ_α is the pointwise closure of Ω_α.*

Proof. The case $\alpha = 1$ is trivial. D. Preiss [7] has shown that each function in Φ_2 is the pointwise limit of a sequence of approximately continuous functions. But the set of approximately continuous functions is a lattice and every approximately continuous function is a Darboux function. Hence from Preiss' result [7] follows the case $\alpha = 2$.

It remains to prove the theorem for $\alpha > 2$. Let $\{I_n\}_{n=1}^{\infty}$ be an enumeration of all open subintervals of I with rational end-points. In Lemma 2 put $J_n = (-\infty, +\infty)$ for each n, to obtain a sequence $\{P_n\}_{n=1}^{\infty}$ of disjoint non-empty nowhere dense perfect sets such that $P_n \subset I_n$. Let g_n be a continuous function defined on P_n which maps this set onto the closed interval $\langle -n, n \rangle$. For each n, let V_n be an operation on the set $\bigcup_{\beta<\alpha} \Phi_\beta$ of functions in Baire classes preceding α such that for each $f \in \bigcup_{\beta<\alpha} \Phi_\beta$, $V_n(f)$ is a function defined as follows:

$$V_n(f)(x) = \begin{cases} g_m(x) & \text{if } x \text{ is in } P_m \text{ with } \quad m \geqq n\,, \\ f(x) & \text{otherwise}\,. \end{cases}$$

Each $V_n(f)$ is in $\bigcup_{\beta<\alpha} \Phi_\beta$. To see it assume that f is in some $\mathscr{B}_\beta$ with $\beta < \alpha$ if α is a finite ordinal and $\beta \leqq \alpha$ otherwise. We assert that $V_n(f)$ is in $\mathscr{B}_{\max(\beta,2)} \subset \bigcup_{\gamma<\alpha} \Phi_\gamma$. Indeed, let λ be a real number. Consider the set

$$[V_n(f) > \lambda] = \bigcup_{i=n}^{\infty} [g_i > \lambda] \cup ([f > \lambda] - \bigcup_{i=n}^{\infty} P_i) .$$

The set $\bigcup_{i=n}^{\infty} [g_i > \lambda]$ is clearly of the type F_σ. The set $([f > \lambda] - \bigcup_{i=n}^{\infty} P_i)$ is adifference of two sets, the first of the additive Borel class β, and the second of the type F_σ. It is easily checked that the difference is in the additive max $(\beta, 2)$. Hence $[V_n(f) > \lambda]$ as the union of two sets, the first of the type F_σ and the second of the additive Borel class max $(\beta, 2)$ is itself of the additive Borel class max $(\beta, 2)$. For $[V_n(f) < \lambda]$ the argument is similar and hence we conclude that $V_n(f) \in \mathscr{B}_{\max(\beta,2)} \subset \bigcup_{\gamma<\alpha} \Phi_\gamma$.

To prove that each $V_n(f)$ is also in $\mathscr{D}$ it suffices to show that $V_n(f)$ takes on each real value on each non-empty open interval J. Let p be a positive integer. J contains some rational open interval I_r with $r > n + p$ hence J contains the set P_r; from the definition of V_n it follows that $V_n(f)(x) = g_r(x)$ for $x \in P_r \subset J$, hence $V_n(f)$ takes on each value $y \in \langle -r, r\rangle \supset \langle -p, p\rangle$ on the interval J.

Since the set $\bigcup_{\beta<\alpha} \Phi_\beta$ is a lattice of functions it follows that for each n, the set $V_n(\bigcup_{\beta<\alpha} \Phi_\beta)$ is a lattice of Darboux functions in $\bigcup_{\beta<\alpha} \Phi_\beta$: Clearly $V_n(\bigcup_{\beta<\alpha} \Phi_\beta) \subset V_{n+1}(\bigcup_{\beta<\alpha} \Phi_\beta)$, for each n. Hence $\Omega_\alpha = \bigcup_{n=1}^{\infty} V_n(\bigcup_{\beta<\alpha} \Phi_\beta)$ is a lattice of Darboux function sand $\Omega_\alpha \subset \bigcup_{\beta<\alpha} \Phi_\beta$.

Finally, let $h \in \Phi_\alpha$. There exists a sequence $\{h_n\}_{n=1}^{\infty}$ of functions in $\bigcup_{\beta<\alpha} \Phi_\alpha$ such that $\lim_{n\to\infty} h_n = h$. It is easy to see that $\lim_{n\to\infty} V_n(h_n) = h$, q.e.d.

Next theorem gives a somewhat sharper result than the above cited Theorems A and B.

Theorem 3. *For each non-limit ordinal $\alpha > 0$*

$$\Phi_\alpha = (\mathscr{D}\Phi_{\alpha-1})\uparrow\downarrow \cap (\mathscr{D}\Phi_{\alpha-1})\downarrow\uparrow .$$

Proof. From the above cited Theorem B it follows that $\Phi_\alpha \supset (\mathscr{D}\Phi_{\alpha-1})\uparrow\downarrow \cap \cap (\mathscr{D}\Phi_{\alpha-1})\downarrow\uparrow$. Thus suppose f to be in Φ_α. By Th. 2 there is a lattice Ω_α of Darboux Baire $\alpha - 1$ functions such that Φ_α is the pointwise closure of Ω_α. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions in Ω_α converging to f. Put

$$g_{n,k} = \max(f_n, f_{n+1}, \ldots, f_{n+k}), \quad h_{n,k} = \min(f_n, f_{n+1}, \ldots, f_{n+k}),$$

and

$$g_n = \sup(f_n, f_{n+1}, \ldots), \quad h_n = \inf(f_n, f_{n+1}, \ldots) .$$

It is easy to see that the functions $g_{n,k}$ and $h_{n,k}$ are in $\mathscr{D}\Phi_{\alpha-1}$, that each g_n is in $(\mathscr{D}\Phi_{\alpha-1})\uparrow$ and each h_n is in $(\mathscr{D}\Phi_{\alpha-1})\downarrow$, and that functions g_n decrease pointwise to f and functions h_n increase pointwise to f, q.e.d.

Remark. If $\alpha > 2$, then the Theorems 2 and 3 can be stated for functions with a more general domain, e.g. for functions defined on a complete separable metric space which is dense in itself. In the proof of Theorem 2 it suffices to replace the rational open intervals $\{I_n\}_{n=1}^{\infty}$ by an open basis $\{G_n\}_{n=1}^{\infty}$, and similarly as in the proof of Lemmas 1 and 2 apply the Alexandroff-Hausdorff theorem (see [4], p. 355) which states that each uncountable Borel set contains a set P which is topologically equivalent to the Cantor set C. Thus the following theorems can be proved:

Theorem 4. *Let X be a complete separable metric space which is dense in itself and let $\alpha > 2$ be an ordinal; there is a lattice Ω_α of real-valued functions defined on X such that $\Omega_\alpha \subset \bigcup_{\beta<\alpha} \Phi_\beta$, each $f \in \Omega_\alpha$ takes on each real value on each non-empty open subset of X, and Φ_α is the pointwise closure of Ω_α.*

Theorem 5. *Let X be a complete separable metric space which is dense in itself and let $\alpha > 2$ be a non-limit ordinal; if $\tilde{\mathscr{D}}\Phi_\beta$ denote the set of all real-valued functions defined on X, in Baire class β, which take on each real value on each non-empty open subset of X then*

$$\Phi_\alpha = (\bigcup_{\beta<\alpha} \tilde{\mathscr{D}}\Phi_\beta)\uparrow\downarrow \bigcup_{\beta<\alpha} \tilde{\mathscr{D}}\Phi_\beta)\downarrow\uparrow .$$

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