Jan Stanisław Lipiński; Tibor Šalát
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ON THE POINTS OF QUASICONTINUITY AND CLIQUISHNESS OF FUNCTIONS

JAN STANISŁAW LIPIŃSKI, Gdańsk and TIBOR ŠALÁT, Bratislava

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Let X, Y be two topological spaces. The function $f : X \to Y$ is said to be quasicontinuous (neighborly) at the point $x_0 \in X$ if for each neighbourhood $U(x_0)$ of the point x_0 (in X) and each neighbourhood $V(f(x_0))$ of the point $f(x_0)$ (in Y) there exists an open set $U \subset U(x_0)$, $U \neq \emptyset$ such that $f(U) \subset V(f(x_0))$ (cf. [1]–[4]).

Let Y be a metric space with the metric ϱ'. The function $f : X \to Y$ is said to be cliquish (apparentée) at the point $x_0 \in X$ if for each $\varepsilon > 0$ and each neighbourhood $U(x_0)$ of the point x_0 (in X) there exists an open set $U \subset U(x_0)$, $U \neq \emptyset$ such that for each two points $y', y'' \in U$ the inequality $\varrho'(f(y'), f(y'')) < \varepsilon$ holds (cf. [3], [4]).

If Y is a metric space and $f : X \to Y$ is quasicontinuous at $x_0 \in X$, then it is cliquish at x_0, too. The converse is not true (cf. [4]).

The function defined on X is said to be quasicontinuous and cliquish (on X), respectively if it is quasicontinuous and cliquish, respectively at each point $x \in X$.

We denote by Q_f (A_f) the set of all such points at which the function f defined on X is quasicontinuous (cliquish). If $f : X \to Y$ and Y is a metric space, then we have $Q_f \subset A_f$.

In this paper we shall study the structure of the sets Q_f, A_f.

Theorem 1. *Let X be a topological space and Y a metric space (with the metric ϱ'). Then A_f is a closed set for each $f : X \to Y$.*

Proof. Let $y_0 \in \overline{A}_f$ ($\overline{M}$ denotes the closure of the set M). We shall prove that $y_0 \in A_f$. Let $U(y_0)$ be an arbitrary neighbourhood of the point y_0. There exists an $x_0 \in A_f$ such that $x_0 \in U(y_0)$. So $U(y_0)$ is a neighbourhood of the point x_0 and since f is cliquish at the point x_0, there exists for $\varepsilon > 0$ an open set $U \subset U(y_0)$, $U \neq \emptyset$ such that $\varrho'(f(y'), f(y'')) < \varepsilon$ holds for each two points $y', y'' \in U$. Following the definition of the cliquishness the function f is cliquish at y_0 and hence $y_0 \in A_f$.

In the paper [3] the following results are proved (Theorems I and II).

Theorem I. *If $f: X \to Y$ (Y is a metric space) is cliquish at each point of a set dense in X, then f is cliquish on X.*

The function $f: X \to Y$ (Y is a metric space) is said to be pointwise cliquish on X if the set $X - A_f$ is a dense set, not closed in X (see [3]).

Theorem II. *The set A_f for an arbitrary pointwise cliquish function f on X is nowhere dense.*

It is easy to see that the both theorems I, II follow at once from the fact that A_f is a closed set (see our Theorem 1).

It is shown in the paper [3] that for each set S nowhere dense in an interval (a, b) there exists such a real function on (a, b) which is cliquish at each point of the set S and is not cliquish at any point of the set $(a, b) - \bar{S}$, $\bar{S}$ being the closure of the set S (relative to (a, b)). We now extend that result of the paper [3].

Theorem 2. *Let X and Y, respectively, be two metric spaces with the metric ϱ and ϱ', respectively. Let us suppose that*

(i) *X has no isolated points;*

(ii) *There exists a one-to-one Cauchy sequence $\{y'_k\}_{k=0}^{\infty}$ of elements of the space Y.*

Then for each set $A \subset X$ closed in X there exists a function $f: X \to Y$ such that $A = A_f$.

Proof. If $A = X$, then we choose an element $y_0 \in Y$ and set $f(x) = y_0$ for each $x \in X$. Evidently $A_f = X$.

Let $A = \emptyset$. Professor W. Sierpiński proved in the paper [5] the following result: If each open non-void set in the metric space M contains at least $m \geqq \aleph_0$ elements, then M can be expressed as a union of m pairwise disjoint sets so that each of these sets has with each open non-void set at least m common points. It follows from this result in view of the assumption (i) of our theorem that X can be decomposed in two disjoint sets X_1, X_2, each X_i ($i = 1, 2$) being dense in X.

Let us choose two points $y_1, y_2 \in Y$, $y_1 \neq y_2$ and define $f(x) = y_i$ for $x \in X_i$ ($i = 1, 2$). Then we have obviously $A_f = \emptyset$.

Let $A \neq \emptyset$, $A \neq X$, $A \subset X$, A be closed in X. Let X_i ($i = 1, 2$) have the previous meaning. We put

$$A_0 = \{x \in X;\ \varrho(x, A) > 1\},$$

$$A_n = \left\{x \in X;\ \frac{1}{n+1} < \varrho(x, A) \leqq \frac{1}{n}\right\} \quad (n = 1, 2, \ldots).$$

Then $X - A = \bigcup_{k=0}^{\infty} A_k$. Let us choose an arbitrary element $y_0 \in Y$ and define the function f in the following way:

$$f(x) = y_0 \quad \text{for} \quad x \in A, \qquad f(x) = y'_{2k} \quad \text{for} \quad x \in X_1 \cap A_k$$

and $f(x) = y'_{2k+1}$ for $x \in X_2 \cap A_k$ $(k = 0, 1, 2, \ldots)$ (see the assumption (ii) of Theorem).

We shall prove that $A_f = A$.

1) Let $x_0 \in A$. If x_0 is an interior point of the set A, then evidently $x_0 \in A_f$. Let us suppose that x_0 is no interior point of the set A. Let $\varepsilon > 0$ and U be a neighbourhood of the point x_0. Let us choose a natural number l such that

$$\text{a)} \quad S(x_0, 1/l) \subset U^{*}) \,; \qquad \text{b)} \quad \operatorname{diam} B_l < \varepsilon \,,$$

where $B_l = \{y'_{2l}, y'_{2l+1}, y'_{2l+2}, \ldots\}$.

Since x_0 is no interior point of A there exists an $x_1 \in S(x_0, 1/l)$ such that $x_1 \notin A$. Since $\varrho(x_1, x_0) < 1/l$ we have $\varrho(x_1, A) < 1/l$. From the continuity of the (real) function $\psi(x) = \varrho(x, A)$ it follows the existence of such a positive real number $\delta_1 > 0$ that

$$\text{a}') \quad S(x_1, \delta_1) \subset X - A \,, \qquad \text{b}') \quad \forall_{x \in S(x_1, \delta_1)} \; \varrho(x, A) < 1/l \,.$$

From this we obtain

$$S_1 = S(x_1, \delta_1) \subset \bigcup_{k=l}^{\infty} A_k \,. \tag{1}$$

Let $x', x'' \in S_1$. From (1) and from the definition of the function f we get $f(x')$, $f(x'') \in B_l$ and hence (see b)) $\varrho'(f(x'), f(x'')) \leqq \operatorname{diam} B_l < \varepsilon$. Hence f is cliquish at x_0.

2) Let $x_0 \in X - A$. Then $\varrho(x_0, A) > 0$. We can find a natural number s such that $\varrho(x_0, A) > 1/(s + 1)$. Following the continuity of the function $\psi(x) = \varrho(x, A)$ there exists a $\delta_0 > 0$ such that for each $x \in S(x_0, \delta_0)$ we have $\varrho(x, A) > 1/(s + 1)$. Hence $S(x_0, \delta_0) \subset \bigcup_{k=0}^{s} A_k$ and so we have

$$f(S(x_0, \delta_0)) \subset \{y'_0, y'_1, \ldots, y'_{2s}, y'_{2s+1}\} = B \,.$$

Let us put $\varepsilon_0 = \min \varrho'(z, u)$, $z, u \in B$, $z \neq u$. Then $\varepsilon_0 > 0$ in view of the assumption (ii) of Theorem. If $S_1 = S(x_1, \delta_1)$ is an arbitrary subsphere of the sphere $S(x_0, \delta_0)$, then $f(S_1) \subset B$ and since $X_i \cap S_1 \neq \emptyset$ $(i = 1, 2)$, the function f attains on S_1 at least one value y'_{2k} for some k, $0 \leqq k \leqq s$ and at least one value y'_{2j+1} for some j, $0 \leqq j \leqq s$. From (ii) we have $\varrho'(y'_{2k}, y'_{2j+1}) \geqq \varepsilon_0$. Hence f is not cliquish at x_0.

Remark 1. The assumption (i) in Theorem 2 is substantial and so it cannot be omitted. If namely X has some isolated point p, then $A = X - \{p\}$ is a closed set. But for each $f : X \to Y$ (Y is an arbitrary metric space) the point p is a point of continuity (and so also of cliquishness) of the function f and hence the equality $A_f = A$ cannot occur for any $f : X \to Y$.

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*) $S(p, \delta) = \{x \in X;\ \varrho(p, x) < \delta\}$ — the spherical neighbourhood of the point p (shortly sphere).

Remark 2. The assumption (ii) in Theorem 2 is certainly fulfilled if Y has at least one accumulation point.

From Theorem 1 and 2 we get at once the following

Theorem 2'. *Let X, Y be two metric spaces satisfying the conditions* (i), (ii) *from Theorem* 2, *let $A \subset X$. Then there exists a function $f : X \to Y$ with $A_f = A$ if and only if A is closed in X.*

In the continuation of this paper we shall restrict ourselves to real functions defined on the n-dimensional Euclidean space R^n. We shall stipulate $X = R^n$ and $Y = R$. For this class of functions the following theorem concerning sets of quasicontinuity points holds:

Theorem 3. *The set E is a set of quasicontinuity points if and only if* $(\operatorname{Int} \bar{E}) - E$ *is a first category set in the sense of Baire.*

Proof. Let f be an arbitrary function defined on the set R^n. Let us denote by Ω_ε the set of all the points x for which there exists a number δ_x such that the conditions $\varrho(x', x) < \delta_x$ and $\varrho(x'', x) < \delta_x$ imply the condition $|f(x') - f(x'')| < \varepsilon$. Ω_ε is evidently open. Let E be the set of quasicontinuity points of f. Let $x_0 \in E$. For an arbitrary number $\eta > 0$ and for the number $\frac{1}{2}\varepsilon$ there exists an interval I such that $I \subset U = S(x_0, \eta)$ and $f(I) \subset (f(x_0) - \frac{1}{2}\varepsilon, f(x_0) + \frac{1}{2}\varepsilon) = V$. It follows from $x' \in I$ and $x'' \in I$ that $|f(x') - f(x'')| < \varepsilon$. We have thus demonstrated that $\operatorname{Int} I \subset \Omega_\varepsilon$. As η was an arbitrary number it follows now that $x_0 \in \Omega'_\varepsilon$, where Ω'_ε denotes the set of accumulation points of Ω_ε. Hence we have shown that $E \subset \Omega'_\varepsilon$. Therefore $\operatorname{Int} \bar{E} \subset \bar{\Omega}_\varepsilon$. Thus the open sets Ω_ε are dense in $\operatorname{Int} \bar{E}$. The set $(\operatorname{Int} \bar{E}) - \Omega_\varepsilon$ is for any $\varepsilon > 0$ nowhere dense. Hence $\bigcup_{n=1}^{\infty} [(\operatorname{Int} \bar{E}) - \Omega_{1/n}] = (\operatorname{Int} \bar{E}) - \bigcap_{n=1}^{\infty} \Omega_{1/n}$ is a first category set. It is sufficient to show that $(\operatorname{Int} \bar{E}) - E \subset (\operatorname{Int} \bar{E}) - \bigcap_{n=1}^{\infty} \Omega_{1/n}$, i.e. that $E \supset \bigcap_{n=1}^{\infty} \Omega_{1/n}$. It can be easy seen that each point of the set $\bigcap_{n=1}^{\infty} \Omega_{1/n}$ is a point of continuity of the function f. Continuity points are always quasicontinuity points. Thus also the last inclusion holds. Hence $(\operatorname{Int} \bar{E}) - E$ is a first category set and we have shown the necessary condition.

Let us assume now that $(\operatorname{Int} \bar{E}) - E$ is a first category set. It is clear that $\bar{E} - \operatorname{Int} \bar{E}$ is nowhere dense. Let $G_0 = \emptyset$ and for $n \geqq 1$ let $G_n = \{x; \varrho(x, \bar{E}) > 1/n\}$. Assume that B_1 is a set dense in $R^n - \bar{E}$ and a border set. Let us accept $f(x) = 0$ for $x \in (R^n - \bar{E}) - B_1$ and $f(x) = 1/n$ for $x \in B_1 \cap (G_n - G_{n-1})$. These conditions define the function f in the set $R^n - \bar{E}$. This function is not quasicontinuous at any point of this set. Indeed, let $x_0 \in R^n - \bar{E}$. Then there exists a number n_0 for which $x_0 \in G_{n_0} - G_{n_0 - 1}$. Let U be the neighbourhood of x_0 enclosed in G_{n_0}. Furthermore assume that $V = (f(x_0) - 1/2n_0, f(x_0) + 1/2n_0)$. The function f attains in U the values 0, $1/n_0$ and possibly $1/(n_0 - 1)$, $1/(n_0 - 2), \ldots, 1$. The value 0 is attained on a set dense in U. Thus the set $f(I)$ must contain (for each interval $I \subset U$) two points distant at least $1/n_0$ from each other. These points cannot belong simultaneously to V. Hence there does not exist an interval $I \subset U$ where $f(I) \subset V$. The function f cannot be quasicontinuous at x_0.

Let us assume now that $f(x) = 0$ for $x \in E - \operatorname{Int} \bar{E}$. We shall demonstrate that f is quasicontinuous at each point of $E - \operatorname{Int} \bar{E}$. Let $x_0 \in E - \operatorname{Int} \bar{E}$. Assume that $U \subset R^n$ and $V \subset R$ are open sets enclosing the points x_0 and $f(x_0)$ respectively. Choose n_0 so that $\langle 0, 1/n_0 \rangle \subset V$. Let I be an arbitrary interval in the set $U \cap \cap \{x; \varrho(x, \bar{E}) < 1/n_0\}$ which is non-void (as x_0 belongs to it) and open. It follows from the definition of f that for $x \in I$ we have $0 \leqq f(x) \leqq 1/n_0$. Thus $f(I) \subset \subset \langle 0, 1/n_0 \rangle \subset V$, what proves the quasicontinuity at x_0.

We shall define now the function f on the set $\operatorname{Int} \bar{E}$. As $(\operatorname{Int} \bar{E}) - E$ is a first category set, there exists a first category set $F \in \mathscr{F}_\sigma$ such that $(\operatorname{Int} \bar{E}) - E \subset F \subset \operatorname{Int} \bar{E}$. We have $F = \bigcup_{n=1}^{\infty} F_n$ where $F_n \subset F_{n+1}$, $F_n = \bar{F}_n$ and the sets F_n are nowhere dense. Assume that $D_n = (\operatorname{Int} \bar{E}) - F_n$. The sets D_n form a decreasing sequence of open sets dense in $\operatorname{Int} \bar{E}$ and $(\operatorname{Int} \bar{E}) \cap E \supset \bigcap_{n=1}^{\infty} D_n = H$. The set H is therefore a G_δ set dense in $\operatorname{Int} \bar{E}$. We choose in $\operatorname{Int} \bar{E}$ a set B_2 which is dense and a border set in $\operatorname{Int} \bar{E}$. Assume now that $f(x) = 0$ for $x \in E \cap \operatorname{Int} \bar{E}$, $f(x) = 1/n$ for $x \in B_2 \cap [(\operatorname{Int} \bar{E}) - E] \cap \cap (D_n - D_{n-1})$ and $f(x) = 1/2n$ for $x \in (R^n - B_2) \cap [(\operatorname{Int} \bar{E}) - E] \cap (D_n - D_{n-1})$. For $x \in D_n$ we have $0 \leqq f(x) \leqq 1/n$. It follows therefrom that f is continuous at each point of H. Assume now that $x_0 \in E \cap \operatorname{Int} \bar{E}$. We choose the sets $U \subset R^n$ and $V \subset R$ such that $x_0 \in U$ and $f(x_0) \in V$. The density of H implies the existence of a point $\bar{x} \in U \cap \operatorname{Int} \bar{E} \cap H$. As $\bar{x}$ is a continuity point of f there exists an interval I such that $\bar{x} \in I$, $I \subset U$ and $f(I) \subset V$. The point x_0 is hence a quasicontinuity point of f.

We shall demonstrate now that the points of $(\operatorname{Int} \bar{E}) - E$ are not quasicontinuity points. The set E is dense in $\operatorname{Int} \bar{E}$. For $x \in E$ we have $f(x) = 0$. The number 0 is enclosed in the set $f(I)$ for each interval $I \subset \operatorname{Int} \bar{E}$. Let $x_0 \in (\operatorname{Int} \bar{E}) - E$. Hence $f(x_0) > 0$. Assume that V is an open set not enclosing 0, such that $f(x_0) \in V$. Then in no neighbourhood U of x_0 enclosed in $\operatorname{Int} \bar{E}$ an interval I exists such that $f(I) \subset V$. This means that there are no qusicontinuity points in $(\operatorname{Int} \bar{E}) - E$.

The function f has not yet been defined in the set $(\bar{E} - \operatorname{Int} \bar{E}) - E$ which is a nowhere dense set. Assume that $f(x) = 2$ for x belonging to this set. In its complement the function f attains values not greater than 1. This complement is dense. We can now easily prove, just as in the case of the set $(\operatorname{Int} \bar{E}) - E$ that a point x in which $f(x) = 2$ cannot be a quasicontinuity point of f.

Finally E is the set of all quasicontinuity points of the function f.

Remark 3. Theorem 3 can be generalized. So for instance the condition formulated in this theorem is necessary for sets Q_f where X is an arbitrary topological space, and Y is a metric space. The condition is also sufficient when X is a complete metric space dense in itself and Y is a metric space possessing at least one accumulation point. The necessity can be shown similarly as the first part of Theorem 3. The sufficiency can be shown by means of methods applied in the proof of Theorem 2 and the second part of the proof of Theorem 3.

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Authors' addresses: Jan Stanisław Lipiński, Instytut Matematyki Uniwersytetu Gdańskiego w Gdańsku, Poland; *Tibor Šalát,* Bratislava, Šmeralova 2, ČSSR (Katedra algebry a teorie čísel Prírodovedeckej fakulty UK).