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ON MULTIPLIERS OF TEMPERATE DISTRIBUTIONS

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In this paper we take a sequence of Banach spaces $H^0 \subset H^{-1} \subset H^{-2} \subset \ldots$ for which $\bigcup_{k \geqq 0} H^{-k} = \mathscr{S}'$, where $\mathscr{S}'$ is the space of temperate distributions. For each pair p, q of non-negative integers we define a normed space $\mathscr{O}_{p,q}$ of multiplication operators from H^{-q} into H^{-p}. Then it appears that $\bigcap_{q \geqq 0} \bigcup_{p \geqq 0} \mathscr{O}_{p,q}$ equals (as a vector space) to the Schwartz's space $\mathscr{O}_M$ of multiplication operators on $\mathscr{S}'$. We get multiplication as a continuous map either from $\mathscr{O}_{p,q} \times H^{-q}$ into H^{-p} or from $\bigcup_{p \geqq 0} \mathscr{O}_{p,q} \times H^{-q}$ into $\mathscr{S}'$. Similar results are shown for the convolution.

Notation. R^n stands for Euclidean n-dimensional space. The set of all non-negative integers is denoted by N, the set of all integers by Z, and the set of all multiindices $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ by N^n. For $x \in R^n$, $\alpha \in N^n$, we write $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$, $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}$, $|x| = (x, x)^{1/2} = (x_1^2 + x_2^2 + \ldots \ldots + x_n^2)^{1/2}$. As multiindices will be always denoted by letters from the beginning of Greek alphabet there should be no confusion of a length $|\alpha|$ with a norm $|x|$, where $\alpha \in N^n$, $x \in R^n$.

C^∞ denotes the vector space of all functions infinitely differentiable on R^n. Those elements from C^∞ which have compact support form a space C_0^∞. The space $\mathscr{S}$ consists of all functions $f \in C^\infty$ which for each pair $\alpha, \beta \in N^n$ fulfil an inequality

$$(1) \qquad q_{\alpha,\beta}(f) = \sup_{x \in R^n} |x^\alpha D^\beta f(x)| < +\infty .$$

Family $\{q_{\alpha,\beta}\}_{\alpha,\beta \in N^n}$ of seminorms defines a locally convex structure on $\mathscr{S}$. Hence the dual $\mathscr{S}'$ exists and its elements are called temperate distributions.

Definition 1. Let us have a set X, a family $\{Y_\iota\}_{\iota \in I}$ of topological spaces, and a family $\{f_\iota\}_{\iota \in I}$ of maps $f_\iota : X \to Y_\iota$. The coarsest topology on X for which all f_ι, $\iota \in I$, are continuous is called *the initial topology on X for the family $\{f_\iota\}_{\iota \in I}$*.

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If, instead of $\{f_\iota\}_{\iota\in I}$, we have a family $\{g_\iota\}_{\iota\in I}$ of maps $g_\iota : Y_\iota \to X$, then the finest topology on X for which all maps g_ι, $\iota \in I$, are continuous is called the *final topology on X for* $\{g_\iota\}_{\iota\in I}$.

We will deal only with families $\{Y_\iota\}_{\iota\in I}$ of locally convex spaces. In this case any initial topology on X is also locally convex and there exists the finest locally convex topology on X for which all g_ι, $\iota \in I$ are continuous. It is called the final locally *convex topology on X* for $\{g_\iota\}_{\iota\in I}$. In the case that $X = \bigcap_{\iota\in I} Y_\iota$, resp. $X = \bigcap_{\iota\in I} Y_\iota$, and all f_ι, resp. g_ι, are identity maps we will simply talk about the *initial (final) topology not mentioning* $\{f_\iota\}_{\iota\in I}(\{g_\iota\}_{\iota\in I})$.

Let us have a family $\{\|.\|_k\}_{k\in N}$ of norms defined on $\mathscr{S}$ and generating the same topology on $\mathscr{S}$ as the family (1). Assume that for each $f \in \mathscr{S}$ we have

$$\|f\|_0 \leqq \|f\|_1 \leqq \|f\|_2 \leqq \dots \tag{2}$$

We denote by H^k the completion of $\mathscr{S}$ with respect to the norm $\|.\|_k$ and by $(H^k)'$ its strong dual with the norm $\|.\|_k'$. Then we have

$$\mathscr{S} \subset \dots \subset H^2 \subset H^1 \subset H^0\,, \quad (H^0)' \subset (H^1)' \subset (H^2)' \subset \dots \subset \mathscr{S}'\,.$$

Moreover, $\mathscr{S} = \bigcap_{k\geqq 0} H^k$ and $\mathscr{S}' = \bigcup_{k\geqq 0} (H^k)'$. We equip $\mathscr{S}'$ with the final locally convex topology generated by $\{(H^k)'\}_{k\in N}$. The original topology on $\mathscr{S}$ equals to the initial topology on $\mathscr{S}$ generated by $\{H^k\}_{k\in N}$ for identity maps.

Proposition 1. *The final localy convex topology on $\mathscr{S}'$ is finer than the bounded convergence topology on $\mathscr{S}'$.*

Proof. Let $B \subset \mathscr{S}$ be bounded in $\mathscr{S}$. As B is bounded in each H^k, $k \in N$, we have $C_k = \sup_{\varphi\in B} \|\varphi\|_k < +\infty$. Hence for each $f \in (H^k)' : \sup_{\varphi\in B} |f(\varphi)| \leqq \|f\|_k' \sup_{\varphi\in B} \|\varphi\|_k = C_k \|f\|_k'$.

Definition 2. For each pair $p, q \in N$ denote by $\mathcal{O}_{p,q}$ the space of all functions u, defined on R^n, for which $v \to uv$ is a continuous map from H^p into H^q. We equip $\mathcal{O}_{p,q}$ with the uniform topology and denote the norm on $\mathcal{O}_{p,q}$ by $\|.\|_{p,q}$.

Proposition 2. *Assume that the convergence in H^0 implies the pointwise convergence almost everywhere in R^n. Then $\mathcal{O}_{p,q}$ is a Banach space.*

Proof. Fix $p, q \in N$. Let $\{u_k\}_{k\in N}$ be a Cauchy sequence in $\mathcal{O}_{p,q}$. Then for each $v \in H^p$, $\{u_k v\}_{k\in N}$ is a Cauchy sequence in H^q and due to (2) also in H^0. As H^q is complete there exists such function $u_v \in H^q$ that $\lim_{k\to\infty} \|u_k v - u_v\|_q = 0$. According to our assumption $\lim_{k\to\infty} u_k(x)\, v(x) = u_v(x)$ for allmost all $x \in R^n$. If we take particularly $v(x) = w(\lambda x)$, where $w \in C_0^\infty$, $w(x) = 1$ for $|x| \leqq 1$, and λ ranges over $(0, 1)$, then we see that for almost all $x \in R^n$ it exists $u(x) = \lim_{k \to \infty} u_k(x)$. Hence $u_v(x) = u(x)\, v(x)$ almost everywhere in R^n.

Finally, $\|uv\|_q = \lim_{k \to \infty} \|u_k v\|_q \leqq \|v\|_p \lim_{k \to \infty} \|u_k\|_{p,q}$, which implies $u \in \mathcal{O}_{p,q}$ and $\lim_{k \to \infty} \|u_k - u\|_{p,q} = 0$.

Recall the definition (see [1], [2], [3]) of the space $\mathcal{O}_M$ of multiplication operators on $\mathcal{S}'$. A function $u \in \mathcal{O}_M$ if and only if $u \in C^\infty$ and for each $\alpha \in N^n$ there exists $k \in N$ such that $\lim_{k \to \infty} (1 + |x|)^{-k} |D^\alpha u(x)| = 0$. A topology in $\mathcal{O}_M$ is defined by a family of seminorms $q_{\varphi,\alpha}(u) = \max_{x \in R^n} |\varphi(x)\, D^\alpha u(x)|$, $\varphi \in \mathcal{S}$, $\alpha \in N^n$.

Theorem 1. $\mathcal{O}_M = \bigcap_{q \geqq 0} \bigcup_{p \geqq 0} \mathcal{O}_{p,q}$.

Proof. 1) $u \in \mathcal{O}_M$. Then $v \to uv$ is a continuous map from $\mathcal{S}$ into $\mathcal{S}$. Hence each $q \in N$ there exists $p_q \in N$ and a constant C_q so that for each $v \in \mathcal{S}$ we have $\|uv\|_q \leqq$ $\leqq C_q \|v\|_{p_q}$. As $\mathcal{S}$ is dense in H^{p_q} the last inequality can be extended for all $v \in H^{p_q}$ which means $u \in \mathcal{O}_{p_q,q}$.

2) $u \in \bigcap_{q \geqq 0} \bigcup_{p \geqq 0} \mathcal{O}_{p,q}$. Then for each $v \in \mathcal{S}$ and $q \in N$ we have $uv \in H^q$. Therefore $uv \in$ $\in \mathcal{S}$. If we particularly take $v(x) = \exp(-|x|^2)$ we get $u(x) = u(x)\, v(x)\, v^{-1}(x)$. But $uv \in \mathcal{S} \subset C^\infty$ and $v^{-1} \in C^\infty$. Thust u is an infinitely differentiable function. Further,

$$\frac{\partial u}{\partial x_1} v = \frac{\partial}{\partial x_1}(uv) - u \frac{\partial v}{\partial x_1} \in \mathcal{S} \quad \text{for each} \quad v \in \mathcal{S}\,.$$

Hence by the mathematical induction we have $(D^\alpha u)\, v \in \mathcal{S}$ for each $\alpha \in N^n$ and $v \in \mathcal{S}$.

Assume that $u \notin \mathcal{O}_M$. Then there exists such $\alpha \in N^n$ that for each $k \in N$ we can find an $x_k \in R^n$ so that $(1 + |x_k|)^{-k} |D^\alpha u(x_k)| \geqq 1$ and $|x_{k+1}| \geqq 2 + |x_k|$. Take a function $w(x) = \exp(|x|^2 - 1)^{-1}$ for $|x| < 1$ and $w(x) = 0$ for $|x| \geqq 1$. Then evidently $v(x) = \sum_{k=0}^{\infty} (1 + |x_k|)^{-k} w(x - x_k) \in \mathcal{S}$ and we have $|D^\alpha u(x_k)\, v(x_k)| \geqq$ $\geqq (1 + |x_k|)^k |v(x_k)| = w(0) > 0$. Hence $(D^\alpha u)\, v \notin \mathcal{S}$ which is a contradiction.

Definition 3. Let $p, q \in N$. For each $u \in \mathcal{O}_{p,q}$, $f \in (H^q)'$, we define uf as an element from $(H^p)'$ by $(uf)\, v = f(uv)$, $v \in H^p$.

Proposition 3. *For each $p, q \in N$ the mapping $(u, f) \to uf$ is continuous from $\mathcal{O}_{p,q} \times (H^q)'$ into $(H^p)'$.*

Proof. By direct calculation we get $\|uf\|'_p \leqq \|u\|_{p,q} \|f\|'_q$.

Definition 4. For each $q \in N$ we write $\mathcal{O}_q = \bigcup_{p \in N} \mathcal{O}_{p,q}$ and $\mathcal{O} = \bigcap_{q \in N} \mathcal{O}_q$. Each space $\mathcal{O}_q$ is equipped with the final locally convex topology and $\mathcal{O}$ with the initial topology.

Proposition 4. *The topology of $\mathcal{O}$ is finer than topology of $\mathcal{O}_M$.*

Proof. Take $\varphi \in \mathcal{S}$, $\varphi \not\equiv 0$, $\alpha \in N^n$, and put $G = \{u \in \mathcal{O}_M;\ q_{\varphi,\alpha}(u) \leqq 1\}$. As the family $\{\|\,.\,\|_k\}_{k \in N}$ of norms on $\mathcal{S}$ is equivalent to (1) there exists $k \in N$ and a constant $C > 0$ so that for each $u \in \mathcal{O}_M$ we have

$$q_{\varphi,\alpha}(u) = \sup_{x \in R^n} |\varphi(x)\, D^\alpha u(x)| \leqq \sum_{\beta+\gamma=\alpha} \sup |D^\beta(uD^\gamma\varphi)| \leqq C \sum_{\gamma \leqq \alpha} \|uD^\gamma\varphi\|_k .$$

According to Theorem 1 for each $u \in \mathcal{O}_M$ there exists $p_u \in N$ such that $u \in \mathcal{O}_{p_u,k}$. Therefore $q_{\varphi,\alpha}(u) \leqq C\|u\|_{p_u,k} \sum_{\gamma \leqq \alpha} \|D^\gamma\varphi\|_{p_u}$. As $\mathcal{O}_{0,k} \subset \mathcal{O}_{1,k} \subset \ldots$ we can choose p_u so that $\sup_{u \in G} p_u = +\infty$. Fix one such mapping $P : u \to p_u$. Then for each $p \in N$ there exists $v \in G$ such that $p_v = \inf\{p_u;\ p_u \geqq p,\ u \in G\}$. We put $G_{p,k} = \{u \in \mathcal{O}_{p,k};\ \|u\|_{p,k} \leqq \leqq (C \sum_{\gamma \leqq \alpha} \|D^\gamma\varphi\|_{p_v})^{-1}$ since for each $u \in \mathcal{O}_{p,k}$ we have $\|u\|_{p,k} \geqq \|u\|_{p+1,k} \geqq \ldots \geqq \|u\|_{p_v,k}$ and $G_{p,k} \subset G_{p+1,k} \subset \ldots \subset G_{p_v,k}$.

Let G_k be the convex hull of $\bigcup_{p \in N} G_{p,k}$. Then G_k is a neighborhood of 0 in $\mathcal{O}_k$. For $u \in G_k$ there are $\lambda_i \geqq 0$, $i = 1, 2, \ldots, m$, $\sum_{i=1}^m \lambda_i = 1$ such that $u = \sum_{i=1}^m \lambda_i u_i$, where $u_i \in G_{p_i,k}$ and all p_i's belong into the range of P. Finally, for $u \in \mathcal{O}_M \cap G_k$ we can write

$$q_{\varphi,\alpha}(u) \leqq \sum_{i=1}^m \lambda_i\, q_{\varphi,\alpha}(u_i) \leqq \sum_{i=1}^m (\lambda_i C \sum_{\gamma \leqq \alpha} \|u_i D^\gamma\varphi\|_k) \leqq$$

$$\leqq \sum_{i=1}^m (\lambda_i C \|u_i\|_{p_i,k} \sum_{\gamma \leqq \alpha} \|D^\gamma\varphi\|_{p_i}) \leqq \sum_{i=1}^m \lambda_i = 1 .$$

Therefore a neighborhood $\mathcal{O} \cap G_k$ of 0 in $\mathcal{O}$ is contained in G which completes the proof.

Theorem 2. *For each $q \in N$ the map $(u, f) \to uf$ is continuous from $\mathcal{O}_q \times (H^q)'$ into $\mathcal{S}'$.*

Proof. Let V be a neighborhood of 0 in $\mathcal{S}'$. Then there is a sequence $\{a_p\}_{p \in N}$ of reals such that V contains the convex hull of $\bigcup_{p \in N} V_p$, where $V_p = \{f \in (H^p)';\ \|f\|'_p \leqq a_p\}$.

For each $p, q \in N$ put $G_{p,q} = \{u \in \mathcal{O}_{p,q};\ \|u\|_{p,q} \leqq a_p\}$, $U_q = \{f \in (H^q)';\ \|f\|'_q \leqq 1\}$. Let G_q be the convex hull of $\bigcup_{p \in N} G_{p,q}$. Then G_q is a neighborhood of 0 in $\mathcal{O}_q$. For $u \in G_{p,q}$, $f \in U_q$ it is $uf \in V_p$ and therefore for $u \in G_q$, $f \in U_q$ we have $uf \in V$.

Remark. We have simultaneously proved that $(u, f) \to uf$ is $\mathcal{O}$-hypocontinuous on $\mathcal{O} \times \mathcal{S}'$, i.e. it is continuous from $B \times \mathcal{S}'$ into $\mathcal{S}'$ for each bounded set $B \subset \mathcal{O}$ and it is continuous from $\mathcal{O} \times f$ into $\mathcal{S}'$ for each $f \in \mathcal{S}'$.

In fact, if $B \subset \mathcal{O}$ is bounded then for each $q \in N$ there is $\lambda_q > 0$ such that $B \subset \lambda_q G_q$. If we put $W_q = \lambda_q^{-1} U_q$ and if W is the convex hull of $\bigcup_{q \in N} W_q$ then W is a neighborhood of 0 in $\mathcal{S}'$ and for $u \in B, f \in W$ we have $uf \in V$.

In the following we take a particular case. Let for $f \in \mathcal{S}$ and $k \in N$

$$\|f\|_k = \left(\sum_{|\alpha| + |\beta| \leqq k} \int_{R^n} |x^\alpha D^\beta f(x)|^2 \, dx \right)^{1/2} . \tag{3}$$

This system of norms on $\mathcal{S}$ is equivalent to the family (1). Each space H^k consists of all functions f which have generalized derivatives $D^\alpha f$ for all $\alpha \in N^n$, $|\alpha| \leqq k$, and $\|f\|_k < +\infty$. As $H^0 = L_2(R^n)$ we can without ambiguity write H^{-k} instead of $(H^k)'$. This notation will simplify our further formulae.

Let $k \in N$. For each $\alpha \in N^n$, $|\alpha| \leqq k$, take a function $g_\alpha \in L_2(R^n)$ and a polynomial P_α of degree $\leqq k - |\alpha|$. Then we have

$$\sum_{|\alpha| \leqq k} P_\alpha D^\alpha g_\alpha \in H^{-k} . \tag{4}$$

On the other hand each distribution $f \in H^{-k}$ can be represented in the form (4). For each $k \in Z$ Fourier transform $\mathcal{F}$ is an automorphism on H^k.

Proposition 5. $\mathcal{O}_{p,q} \subset H^{q-p-r}$, *where* $p, q \in N$ *and* $r = 1 + [\frac{1}{2}n]$.

Proof. Take $u \in \mathcal{O}_{p,q}$ and put $v = (1 + |x|^2)^{-(p+r)/2}$. Then $v \in H^p$ and $uv \in H^q$. For each $k \in Z$ the map $\Phi : f \to (1 + |x|^2)^{1/2} f$ evidently maps H^k into H^{k-1}. Hence $u = \Phi^{p+r}(uv) \in H^{q-p-r}$.

Definition 5. For each pair $p, q \in N$ we write $\mathcal{O}^*_{p,q} = \mathcal{F}\mathcal{O}_{p,q}$, $\mathcal{O}^*_q = \mathcal{F}\mathcal{O}_q$, $\mathcal{O}^* = \mathcal{F}\mathcal{O}$, where $\mathcal{F}$ stands for Fourier transform. We define a topology in $\mathcal{O}^*_{p,q}$ by a norm $\|f\|^*_{p,q} = \|\mathcal{F}^{-1} f\|_{p,q}$, where $f \in \mathcal{O}^*_{p,q}$, and in $\mathcal{O}^*_q = \bigcup_{p \in N} \mathcal{O}^*_{p,q}$, resp. $\mathcal{O}^* = \bigcap_{q \in N} \mathcal{O}^*_q$, as the final locally convex, or initial, topology.

Due to Proposition 5 the definition of spaces $\mathcal{O}^*_{p,q}$ is meaningful. It follows from Proposition 2 that each $\mathcal{O}^*_{p,q}$ is a Banach space and therefore the definition of topologies in $\mathcal{O}^*_q$ and $\mathcal{O}^*$ is also all right.

The Schwartz's space $\mathcal{O}'_C$ of convolution operators (see [1], [2], [3]) can be defined as $\{f \in \mathcal{S}'; \mathcal{F}^{-1} f \in \mathcal{O}_M\}$ with such topology that Fourier transform $\mathcal{F} : \mathcal{O}_M \to \mathcal{O}'_C$ is an isomorphism. Thus as an immediate consequence of Theorem 1, we get

Theorem 3. $\mathcal{O}'_C = \bigcap_{q \in N} \bigcup_{p \in N} \mathcal{O}^*_{p,q}$. *The topology of the right side is finer than the one of the left side.*

Proposition 6. *For $u \in \mathcal{O}^*_{p,q}$, $f \in H^{-q}$, the convolution $u * f$, defined by $(u * f)\, v = (u_x \otimes f_y)\, v(x + y)$, where $v \in H^p$, has sense and the map $(u, f) \to u * f$ is continuous from $\mathcal{O}^*_{p,q} \times H^{-q}$ into H^{-p}.*

Proof. Let $u \in \mathcal{O}^*_{p,q}$, $f \in H^{-q}$, $\varphi \in \mathcal{S}$. As $\mathcal{F}^{-1}u \in \mathcal{O}_{p,q}$ is a function we can write $u_x(\varphi(x + y)) = (\mathcal{F}^{-1}u_x)\,(\mathcal{F}_x\varphi(x + y)) = (\mathcal{F}^{-1}u_x)\,((\mathcal{F}\varphi)\,(x) \exp(2\pi i x, y)) = \mathcal{F}^{-1}(\mathcal{F}^{-1}u \,.\, \mathcal{F}\varphi)$. Mapping $v \to \mathcal{F}^{-1}(\mathcal{F}^{-1}u \,.\, \mathcal{F}v)$ is composed of three continuous maps $v \to \mathcal{F}v \to \mathcal{F}^{-1}u \,.\, \mathcal{F}v \to \mathcal{F}^{-1}(\mathcal{F}^{-1}u \,.\, \mathcal{F}v)$ of $H^p \to H^p \to H^q \to \to H^q$. Hence $f(\mathcal{F}^{-1}(\mathcal{F}^{-1}u \,.\, \mathcal{F}v)$ has sense and it represents a distribution from H^{-p}. Finally, $f(\mathcal{F}^{-1}(\mathcal{F}^{-1}u \,.\, \mathcal{F}v)) = f_y(u_x(v(x + y))) = (u_x \otimes f_y)\, v(x + y) = (u * f)\, v$.

Similarly as in Theorem 2 we can prove that for each $q \in N : (u, f) \to u * f$ maps continuously $\mathcal{O}^*_q \times H^{-q}$ into $\mathcal{S}'$ and it is $\mathcal{O}^*$-hypocontinuous on $\mathcal{O}^* \times \mathcal{S}'$.

Proposition 7. *Let $p, q \in N$, $p < q$. Then $\mathcal{O}_{p,q} = \{0\}$.*

Proof. We show at first that for each $u \in \mathcal{O}_{p,q}$ and $x_0 \in R^n$ we have

$$\lim_{\varrho \to 0+} \operatorname*{ess\,sup}_{|x - x_0| \leqq \varrho} \sum_{|\alpha| \leqq q} |D^\alpha u(x)| < +\infty \,. \tag{5}$$

In fact, for $v(x) = \exp(-|x|^2)$ we get $uv \in H^q$ which implies (5) for the function uv. But then (5) must hold also for the function $u = uv \,.\, v^{-1}$.

Take $u \in \mathcal{O}_{p,q}$ and assume that $M_0 = \{x \in R^n;\ u(x) \neq 0\}$ has positive Lebesgue measure $\mu(M_0)$. There exists $M \subset M_0$ such that $\mu(M) > 0$ and u is continuous on M. Take a point $x_0 \in M$ such that for $B = \{x \in R^n;\ |x - x_0| \leqq 1\}$ we have $\mu(M \cap B) > 0$.

Now, take such $v \in C_0^\infty$ that $\operatorname{supp} v = B$ and put $v_\lambda(x) = v(x_0 + (x - x_0)/\lambda)$, where $\lambda > 0$. Using a substitution $x = x_0 + \lambda(y - x_0)$ we can write

$$\|uv_\lambda\|_q^2 \leqq \|u_\lambda\|_{p,q}^2 \|v_\lambda\|_p^2 = \|u\|_{p,q}^2 \sum_{|\alpha| + |\beta| \leqq p} \int_{R^n} |(x_0 + \lambda(y - x_0))^\alpha \lambda^{-|\beta|} D_y^\beta v(y)|^2 \lambda^n \,\mathrm{d}y \,.$$

Hence

$$\limsup_{\lambda \to 0+} \lambda^{2p-n} \|uv_\lambda\|_q^2 \leqq \|u\|_{p,q}^2 \sum_{|\beta| = p} \int_{R^n} |D^\beta v(y)|^2 \,\mathrm{d}y < +\infty \,.$$

On the other hand

$$\|uv_\lambda\|_q \geqq \left(\int_{R^n} \left|\frac{\partial^q}{\partial x_1^q}(uv_\lambda)\right|^2 \mathrm{d}x\right)^{1/2} \geqq \left(\int_{R^n} \left|u \frac{\partial^q v_\lambda}{\partial x_1^q}\right|^2 \mathrm{d}x\right)^{1/2} - \left(\sum_{i=1}^q \binom{q}{i} \int_{R^n} \left|\frac{\partial^i u}{\partial x_1^i} \frac{\partial^{q-i} v_\lambda}{\partial x_1^{q-i}}\right|^2 \mathrm{d}x\right)^{1/2},$$

$$\liminf_{\lambda \to 0+} \lambda^{2q-n} \int_{R^n} \left|u \frac{\partial^q v_\lambda}{\partial x_1^q}\right|^2 \mathrm{d}x = \liminf_{\lambda \to 0+} \int_{R^n} \left|u(x_0 + \lambda(y - x_0)) \frac{\partial^q v(y)}{\partial y_1^q}\right|^2 \mathrm{d}y \geqq$$

$$\geqq \liminf_{\lambda \to 0+} \int_{B \cap M} \left|u(x_0 + \lambda(y - x_0)) \frac{\partial^q v(y)}{\partial y_1^q}\right|^2 \mathrm{d}y = |u(x_0)|^2 \int_{B \cap M} \left|\frac{\partial^q v(y)}{\partial y_1^q}\right|^2 \mathrm{d}y = A > 0 \,.$$

Due to (5) there exists such $\varrho \in (0, 1)$ that $\operatorname{ess\,sup}_{|x-x_0| \leqq \varrho} \sum_{|\alpha| \leqq q} |D^\alpha u(x)| = C < +\infty$. Thus for $\lambda \in (0, \varrho)$ we have

$$\int_{R^n} \left| \frac{\partial^i u}{\partial x_1^i} \frac{\partial^{q-i} v_\lambda}{\partial x_1^{q-i}} \right|^2 \mathrm{d}x = \lambda^{n-2q} \int_B \left| \frac{\partial^i u(x_0 + \lambda(y - x_0))}{\partial y_1^i} \frac{\partial^{q-i} v(y)}{\partial y_1^{q-i}} \right|^2 \mathrm{d}y \leqq$$

$$\leqq \lambda^{n-2q+2i} C^2 \int_B \left| \frac{\partial^{q-i} v(y)}{\partial y_1^{q-i}} \right|^2 \mathrm{d}y .$$

Summing up we get a desired contradiction

$$0 < A \leqq \liminf_{\lambda \to 0+} \lambda^{2q-n} \left(\|u v_\lambda\|_q^2 + \sum_{i=1}^{q} \binom{q}{i} \int_{R^n} \left| \frac{\partial^i u}{\partial x_1^i} \frac{\partial^{q-i} v_\lambda}{\partial x_1^{q-i}} \right|^2 \mathrm{d}x \right) \leqq$$

$$\leqq \liminf_{\lambda \to 0+} C^2 \sum_{i=1}^{q} \binom{q}{i} \lambda^{2i} \int_B \left| \frac{\partial^{q-i} v(y)}{\partial y_1^{q-i}} \right|^2 \mathrm{d}y = 0 .$$

Proposition 8. *Let a function u be defined on R^n and has generalized derivatives of all orders $\leqq q$. Let there be such $s \in N$ that*

$$\sigma_{q+s,q}(u) = \sum_{|\alpha| \leqq q} \operatorname{ess\,sup}_{x \in R^n} (1 + |x|^2)^{-(s+|\alpha|)/2} |D^\alpha u(x)| < +\infty .$$

Then $u \in \mathcal{O}_{q+s,q}$. Moreover, it exists a constant $C > 0$ (which does not depend on u) for which

$$\|u\|_{q+s,q} \leqq C \, \sigma_{q+s,q}(u) . \tag{6}$$

Proof by direct calculation.

It is well known that if $u \in \mathcal{O}_{0,0}$ then $\sigma_{0,0}(u) < +\infty$. This may not be the case for any space $\mathcal{O}_{p,q}$. It was shown in [5] that in one-dimensional case $\sup_{(-\infty,\infty)} |v(x)| \leqq \|v\|_1$ for each $v \in H^1$. Hence for $u \in H^0$ we have $\|uv\|_0^2 = \int_{-\infty}^{\infty} |uv|^2 \, \mathrm{d}x \leqq \|v\|_1^2 \|u\|_0^2$ which means $u \in \mathcal{O}_{1,0}$. Thus we can easily find an element $u \in L^2(-\infty, \infty) = H^0$ for which $\sigma_{1,0}(u) = +\infty$.

Proposition 9. *If $u \in \mathcal{O}_{p,q}$ then u has generalized derivatives of all orders $\leqq q$.*

Proof is contained in [5].

Example. Derivatives of a function $u \in \mathcal{O}_{p,q}$ need not be continuous on R^n. Take $n = 1$ and put $u(x) = (1 + |x|)^{-1/2}$. Then $u \in \mathcal{O}_{1,1}$ and $\mathrm{d}u/\mathrm{d}x$ is not continuous on R^n. However it is shown in [5] that the space C^k of all k-times continuously differentiable functions contains H^{k+r}, where $r = 1 + [\frac{1}{2}n]$. Thus if $u \in \mathcal{O}_{p+r,q+r}$ then $u \exp(-|x|^2) \in H^{q+r} \subset C^k$ and again $u = (u \exp(-|x|^2) \exp |x|^2 \in C^k$.

Definition 6. For each pair $p, q \in N$ denote by $\mathscr{P}_{p,q}$ the vector space $\{u \in \mathscr{O}_{p,q};\ \sigma_{p,q}(u) < +\infty\}$ with a norm $\sigma_{p,q}$. Further, put $\mathscr{P}^*_{p,q} = \mathscr{F}\mathscr{P}_{p,q}$ with a norm $\sigma^*_{p,q}(u) = \sigma_{p,q}(\mathscr{F}^{-1}u)$, where $u \in \mathscr{P}^*_{p,q}$.

For each $q \in N$ we also define $\mathscr{P}_q = \bigcup_{p\in N} \mathscr{P}_{p,q}$ and $\mathscr{P} = \bigcap_{q\in N} \mathscr{P}_q$; $\mathscr{P}^*_* = \bigcup_{p\in N} \mathscr{P}^*_{p,q}$ and $\mathscr{P}^* = \bigcap_{q\in N} \mathscr{P}^*_q$. We equip these spaces with the final locally convex and initial topologies.

As evidently each $\mathscr{P}_{p,q}$, $\mathscr{P}^*_{p,q}$, with the norm $\sigma_{p,q}$, $\sigma^*_{p,q}$ respectively, is a Banach space all topologies defined in Definition 6 have sense.

Theorem 3. 1) $\mathscr{O}_M = \mathscr{P}$, *and* $\mathscr{O}'_C = \mathscr{P}^*$. *Both these equalities are meant as between vector spaces. The spaces on the right side have finer topology than the corresponding left side.*

2) *The mappings* $(u, f) \to uf$, $(u, f) \to u * f$, *are continuous from* $\mathscr{P}_{p,q} \times H^{-q}$ *into* H^{-p}, $\mathscr{P}^*_{p,q} \times H^{-q}$ *into* H^{-p}, *respectively, for each pair* $p, q \in N$, *and they are also continuous respectively from* $\mathscr{P}_q \times H^{-q}$, $\mathscr{P}^*_q \times H^{-q}$, *into* $\mathscr{S}'$, *for each* $q \in N$.

Proof. 1) We have only to show $\mathscr{O}_M \subset \bigcap_{p\in N} \bigcup_{q\in N} \mathscr{P}_{p,q}$. Take $u \in \mathscr{O}_M$, then according to the definition of $\mathscr{O}_M$ for each $q \in N$ there exists such $p_q \in N$ that $u \in \mathscr{P}_{p_q,q}$.

2) The continuity follows from (6).

Lars Hörmander is using in [4] some spaces of temperate distributions which he denotes by $\mathscr{B}_{p,k}$, where $1 \leqq p \leqq +\infty$ and k is a positive function, defined on R^n, for which there are constants $C > 0$, $s \in R$, such that

$$(7)\qquad k(x + y) \leqq (1 + C|x|)^s\, k(y)\,, \quad x, y \in R^n\,.$$

A temperate distribution u belongs into $\mathscr{B}_{p,k}$ if and only if $\mathscr{F}u$ is a function and $\|u\|_{p,k} = (\int_{R^n} |k \,.\, \mathscr{F}u|^p \,\mathrm{d}x)^{1/p} < +\infty$. If $p = +\infty$ then $\|u\|_{\infty,k} = \operatorname{ess\,sup}_{x\in R^n} |k(x)\, \mathscr{F}u(x)|$.

There is a relation between spaces $\mathscr{P}^*_{p,q}$ and $\mathscr{B}_{\infty,k_s}$, where $k_s = (1 + |x|^2)^{s/2}$.

Proposition 9. $u \in \mathscr{P}^*_{p,q}$ *if and only if* $x^\alpha u \in \mathscr{B}_{\infty,k_{q-p-|\alpha|}}$ *holds for each* $\alpha \in N^n$, $|\alpha| \leqq q$. *Moreover,*

$$\sigma^*_{p,q}(u) = \sum_{|\alpha|\leqq q} \|(2\pi x)^\alpha u\|_{\infty,k_{q-p-|\alpha|}}\,, \quad u \in \mathscr{P}^*_{p,q}\,.$$

Proof. Take $u \in \mathscr{P}^*_{p,q}$. Then

$$\sigma^*_{p,q}(u) = \sigma_{p,q}(\mathscr{F}^{-1}u) = \sigma_{p,q}(\mathscr{F}u) = \sum_{|\alpha|\leqq q} \operatorname{ess\,sup}_{x\in R^n} (1 + |x|^2)^{(q-p-|\alpha|)/2}\, |D^\alpha(\mathscr{F}u)(x)| =$$
$$= \sum_{|\alpha|\leqq q} \operatorname{ess\,sup}_{x\in R^n} k_{q-p-|\alpha|}(x)\, |\mathscr{F}((-2\pi i x)^\alpha u(x))| = \sum_{|\alpha|\leqq q} \|(2\pi x)^\alpha u\|_{\infty,k_{q-p-|\alpha|}}\,.$$

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