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# GROUPS AND HOMOGENEOUS GRAPHS

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In this paper we shall give a construction of some types of homogeneous and weakly homogeneous graphs (directed or undirected). The concept of the homogeneous graph was introduced by M. FIEDLER and V. KNICHAL. We shall give the definition (for undirected graphs).

A homogeneous (undirected) graph is a graph $G$ having the following two properties:

(α) To any two vertices $u, v$ of $G$ there exists an automorphism $\varphi$ of $G$ such that $\varphi(v) = v$,

(β) For any vertex $v$ of $G$ and any permutation $\pi$ of the set of edges incident with $v$ there exists an automorphism $\psi$ of $G$ such that $\psi(v) = v$ and the permutation $\pi$ is induced by $\psi$.

In [2] the concept of a weakly homogeneous graph was introduced.

A weakly homogeneous (undirected) graph is a graph $G$ satisfying (α) and the following condition (weaker than (β)):

(β′) For any vertex $v$ of $G$ and any two edges $e_1, e_2$ incident with it there exists an automorphism $\psi$ of $G$ such that $\psi(v) = v$, $\psi(e_1) = e_2$.

Finally, a graph satisfying (α) will be called symmetric (following L. LOVÁSZ). At first we shall investigate symmetric graphs in general.

Let $v$ be a vertex of a symmetric undirected graph $G$. Let $w_1, \ldots, w_k$ be the vertices joined by an edge with $v$ and let $\varphi_i$ for each $i$ be an automorphism of $G$ such that $\varphi_i(v) = w_i$. Let $\mathfrak{G}$ be the subgroup of the group of all automorphisms of $G$ which is generated by the mappings $\varphi_i$. We shall prove

**Theorem 1.** *Let $G_0$ be the subgraph of $G$ generated by all vertices $\psi(v)$, where $\psi \in \mathfrak{G}$. Then $G_0$ is a connected component of $G$.*

Proof. Let $\psi \in \mathfrak{G}$ and let $\psi = \psi_1\psi_2 \ldots \psi_r$, where each $\psi_i$ for $i = 1, \ldots, r$ is equal to some $\varphi_j$ for $j = 1, \ldots, k$ or to its inverse. Consider the sequence of vertices

$v, \psi_1(v), \psi_1\psi_2(v), \ldots, \psi_1 \ldots \psi_r(v)$. Any two neighbouring elements of this sequence (except the first two which are evidently joined by an edge) are of the form $\psi_1 \ldots$ $\ldots \psi_i(v), \psi_1 \ldots \psi_{i+1}(v)$ for $1 \leqq i \leqq r - 1$. This pair is the image of the pair $v, \psi_{i+1}(v)$ in the automorphism $\psi_1\psi_2 \ldots \psi_i$. If $\psi_{i+1} = \varphi_j$ for some $j$, $1 \leqq j \leqq k$, then $v\psi_{i+1}(v)$ is an edge of $G$ and therefore also $\psi_1 \ldots \psi_i(v)\, \psi_1 \ldots \psi_{i+1}(v)$ is an edge of $G$. If $\psi_{i+1} = \varphi_j^{-1}$ for some $j$, $1 \leqq j \leqq k$, then $v\, \psi_{i+1}(v)$ is an image of the edge $v\, \varphi_j(v)$ in the mapping $\varphi_j^{-1}$ and therefore it is also an edge of $G$. We have proved that there exists a path between $v$ and $\psi(v)$ for each $\psi \in \mathfrak{G}$, all of whose vertices are images of $v$ in mappings of $\mathfrak{G}$, thus the subgraph $G_0$ of $G$ generated by all $\psi(v)$ for $\psi \in \mathfrak{G}$ is connected. Now let $x$ be a vertex of $G$ which cannot be expressed by $\psi(v)$ for any $\psi \in \mathfrak{G}$ and assume that $x$ is joined by an edge with some $\psi_0(v)$, where $\psi_0 \in \mathfrak{G}$. As $G$ is symmetric, we have $x = \chi(v)$, where $\chi$ is some automorphism of $G$ not belonging to $G$. Take the automorphism $\psi_0^{-1}$. The image of $x$ in $\psi_0^{-1}$ is $\psi_0^{-1}\, \chi(v)$, the image of $\psi_0(v)$ is $v$. Thus $v$ is joined by an edge with $\psi_0^{-1}\, \chi(v)$ and $\psi_0^{-1}\, \chi(v) = \varphi_i(v)$ for some $i$, $1 \leqq i \leqq k$. But then $\chi(v) = \psi_0\, \varphi_i(v)$. Both $\psi_0$ and $\varphi_i$ are in $\mathfrak{G}$, therefore also their product $\psi_0\mathfrak{F}_i$ is in $\mathfrak{G}$. This is a contradiction with the assumption that $x$ cannot be expressed as an image of $v$ in a mapping of $\mathfrak{G}$. Thus $G_0$ is a connected component of $G$, q.e.d.

**Theorem 2.** *Let the group of all automorphisms of G be Abelian. Let $\psi_1(v) = \psi_2(v)$ for some elements $\psi_1, \psi_2$ of this group. Then $\psi_1 \equiv \psi_2$.*

Proof. As $\psi_1(v) = \psi_2(v)$, we have $\psi_1^{-1}\, \psi_2(v) = v$. Any vertex $x$ of $G$ can be expressed as $\chi(v)$, where $\chi$ is some automorphism of $G$. Thus $\psi_1^{-1}\, \psi_2(x) = \psi_1^{-1}\psi_2\, \chi(v) = \chi\psi_1^{-1}\, \psi_2(v) = \chi(v) = x$. As $x$ was chosen quite arbitrarily, $\psi_1^{-1}\psi_2$ is an identical mapping and $\psi_1 \equiv \psi_2$.

Now let $\mathfrak{G}$ be a group, $A$ its subset. The graph $G(\mathfrak{G}, A)$ is defined as follows: The vertices of $G(\mathfrak{G}, A)$ are elements of $\mathfrak{G}$ and there exists an edge joining $x$ and $y$ in $G(\mathfrak{G}, A)$ if and only if either $x^{-1}y$ or $y^{-1}x$ is in $A$. The investigation of these graphs was suggested by J. Jakubík.

The preceding theorems show that any symmetric graph is a homomorphic (even isomorphic in the case when $\mathfrak{G}$ is Abelian) image of some $G(\mathfrak{G}, A)$. It is connected if and only if $A$ is a system of generators of $\mathfrak{G}$.

**Theorem 3.** *The graph $G(\mathfrak{G}, A)$ is symmetric.*

Proof. If $u, v$ are two vertices of $G(\mathfrak{G}, A)$, we take a mapping $\varphi_{vu^{-1}}$ such that $\varphi_{vu^{-1}}(a) = vu^{-1}a$ for any $a \in \mathfrak{G}$. The mapping $\varphi_{vu^{-1}}$ is an automorphism of $G(\mathfrak{G}, A)$; we have

$$[\varphi_{vu^{-1}}(x)]^{-1}\, \varphi_{vu^{-1}}(y) = (vu^{-1}x)^{-1}\, vu^{-1}y = x^{-1}uv^{-1}vu^{-1}y = x^{-1}y\,,$$

$$[\varphi_{vu^{-1}}(y)]^{-1}\, \varphi_{vu^{-1}}(x) = (vu^{-1}y)^{-1}\, vu^{-1}x = y^{-1}uv^{-1}vu^{-1}x = y^{-1}x$$

and thus the existence of the edge $\varphi_{vu^{-1}}(x)\,\varphi_{vu^{-1}}(y)$ is equivalent to the existence of the edge $xy$. The automorphism $\varphi_{vu^{-1}}$ evidently maps $u$ onto $v$.

Now we shall investigate homogeneous graphs.

**Theorem 4.** *Let $\varphi$ be an automorphism of the group $\mathfrak{G}$ such that $\varphi(A^*) = A^*$, where $A^*$ is the set of all elements of $A$ and their inverses. Then $\varphi$ is also an automorphism of $G(\mathfrak{G}, A)$.*

Proof. Two vertices $x$, $y$ of $G(\mathfrak{G}, A)$ are joined by an edge if and only if either $x^{-1}y$ or $y^{-1}x$ is in $A$. This occurs if and only if $x^{-1}y$ is in $A^*$. As $\varphi$ is an automorphism of $G$, we have

$$[\varphi(x)]^{-1}\,\varphi(y) = \varphi(x^{-1})\,\varphi(y) = \varphi(x^{-1}y)\,.$$

As $\varphi(A^*) = A^*$ and $\varphi$ is one-to-one, we see that $\varphi(x^{-1}y) \in A^*$ if and only if $x^{-1}y \in \in A^*$. We see that the vertices $\varphi(x)$, $\varphi(y)$ are joined by an edge if and only if the vertices $x$, $y$ are joined by an edge. Therefore $\varphi$ is also an automorphism of $G(\mathfrak{G}, A)$.

**Theorem 5.** *Let $\mathfrak{G}$ be a group, $A$ a system of its generators, $A^*$ the set of all elements of $A$ and their inverses. Let any permutation of $A^*$ be induced by some automorphism of $\mathfrak{G}$. Then $G(\mathfrak{G}, A)$ is a connected homogeneous graph.*

Proof. The property ($\alpha$) was already proved. Take the unit element $e$ of $\mathfrak{G}$. The element $e$ is joined by edges with all elements of $A^*$ and only with them. Thus if $\pi$ is some permutation of $A^*$, we have an automorphism $\psi_\pi$ of $\mathfrak{G}$ which induces $\pi$. According to Theorem 4, $\psi_\pi$ is also an automorphism of $G(\mathfrak{G}, A)$ and the property ($\beta$) is fulfilled for the vertex $e$. Let $x$ be some other vertex and let $\pi$ be some permutation of the set of vertices incident with it (we may evidently use this permutation instead of a permutation of edges incident with $x$). These elements are exactly all $xa$ for $a \in A^*$. Let $\tilde{\pi}$ be the permutation of $A^*$ such that $\tilde{\pi}(a_1) = a_2$ for $a_1 \in A$, $a_2 \in A$ if and only if $\pi(xa_1) = xa_2$. The permutation $\tilde{\pi}$ is induced by an automorphism $\psi_{\tilde{\pi}}$ of $\mathfrak{G}$. Now we take the autmorphism $\varphi_x \psi_{\tilde{\pi}} \varphi_{x^{-1}}$ (where $\varphi_x$ for $x \in \mathfrak{G}$ was defined in the proof of Theorem 3); this is evidently the required automorphism of $G(\mathfrak{G}, A)$.

Remark. The assumption that any permutation of $A^*$ is induced by some automorphism of $\mathfrak{G}$ can be satisfied only if either $A^*$ consists of two elements inverse to each other (the group $\mathfrak{G}$ is cyclic), or all elements of $A$ are of the order 2 (this means $A = A^*$). Otherwise there would exist two different elements $a_1$, $a_2$ of $A$ of an order greater than two, thus $a_1$, $a_2$, $a_1^{-1}$, $a_1^{-1}$ would be four pairwise different elements. Any permutation which lets $a_1$ fixed and maps $a_1^{-1}$ onto $a_2$ cannot be induced by any automorphism of $\mathfrak{G}$, because it maps the pair of mutually inverse elements onto the pair of elements which are not inverse to each other.

Let us take the free group with the free generators $a_1, \ldots, a_k$ and defining relations $a_i^2 = e$ for $i = 1, \ldots, k$ (where $e$ is the unit element of the group). The resulting group will be denoted by $\mathfrak{F}^{(2)}$. According to a theorem due to Dyck [1] any group with $k$ independent generators, $k > 1$, satisfying the assumption of Theorem 5 is a factor-group of $\mathfrak{F}^{(2)}$. It can be derived from it by a system of defining relations which is symmetric with respect to all $a_1, \ldots, a_k$ (i.e. an arbitrary permutation of these symbols does not change the system of defining relations). With help of these groups we can construct various homogeneous graphs.

**Theorem 6.** *Let $\mathfrak{G}$ be an Abelian group, $A$ a system of its generators. To any two elements $a_1, a_2$ of $A$ let there exist an automorphism $\varphi$ of $\mathfrak{G}$ such that $\varphi(A) = A$, $\varphi(a_1) = a_2$. Then $G(\mathfrak{G}, A)$ is a connected weakly homogeneous graph.*

Proof. As $\mathfrak{G}$ is Abelian, there exists an automorphism $\alpha$ of $\mathfrak{G}$ such that $\alpha(x) = x^{-1}$ for each $x \in \mathfrak{G}$. Therefore, if $\varphi$ is an automorphism of $\mathfrak{G}$ mapping some $a_1 \in A$ onto $a_2 \in A$, the automorphism $\alpha\varphi$ maps $a_1$ onto $a_2^{-1}$ and evidently $\alpha\varphi(A^*) = A^*$ (where $A^*$ was defined in Theorem 4). Therefore to any two elements of $A^*$ there exists an automorphism mapping one of them onto the other and mapping $A^*$ onto $A^*$. This automorphism is an automorphism of $G(\mathfrak{G}, A)$. We have proved the property ($\beta'$) for the vertex $e$. Further we proceed as in the proof of Theorem 5.

Now we shall use the obtained results for the construction of a certain class of weakly homogeneous graphs. Let $\mathfrak{A}_1, \ldots, \mathfrak{A}_k$ be cyclic groups of the same order $r$, let $a_i$ be the generator of $\mathfrak{A}_i$ for $i = 1, \ldots, k$. Let $\mathfrak{G}$ be the direct product of $\mathfrak{A}_1, \ldots, \mathfrak{A}_k$, let $A = \{a_1, \ldots, a_k\}$. Evidently the graph $G(\mathfrak{G}, A)$ is weakly homogeneous. This graph is uniquely determined by $k$ and $r$ (up to an isomorphism), therefore we can denote it by $WHG(k, r)$. The numbers $k$, $r$ are positive integers, $r \geqq 2$. According to the above proved results, $WHG(k, r)$ is homogeneous if and only if either $k = 1$ or $r = 2$. The graph $WHG(1, r)$ for $r > 2$ is a circuit $G_r$ of the length $r$, for $r = 2$ it is the graph consisting of one edge and its terminal vertices. The graph $WHG(k, 2)$ is the graph $Q_k$ of the $k$-dimensional cube. The graph $WHG(2, 3)$ is on Fig. 1.

We can generalize this concept also for infinite graphs. For $k$ we can takc any cardinal number, for $r$ at most $\aleph_0$ (because a cyclic group cannot have an order greater than $\aleph_0$). The graph $WHG(k, \aleph_0)$, where $k$ is a positive integer is the lattice graph in the $k$-dimensional space.

The graphs described in [2] are also $G(\mathfrak{G}, A)$, where $\mathfrak{G}$ is a cyclic group and $A$ is the set of generators of $\mathfrak{G}$ such that the assumption of Theorem 6 is satisfied.

Now we shall study directed graphs (digraphs).

A homogeneous digraph is a digraph $G$ having the following three properties [3]:

($\alpha$) To any two vertices $u$, $v$ of $G$ there exists an automorphism $\varphi$ of $G$ such that $\varphi(u) = v$.

($\beta$) For any vertex $v$ of $G$ and any permutation $\pi$ of the set of edges outgoing from $v$ there exists an automorphism $\psi$ of $G$ such that $\psi(v) = v$ and the permutation $\pi$ is induced by $\psi$.

($\gamma$) For any vertex $v$ of $G$ and any permutation $\pi'$ of the set of edges incoming into $v$ there exists an automorphism $\psi'$ of $G$ such that $\psi'(v) = v$ and the permutation $\pi'$ is induced by $\psi'$.

A weakly homogeneous digraph is a digraph $G$ satisfying ($\alpha$) and the following two conditions:

($\beta'$) For any vertex $v$ of $G$ and any two edges $e_1$, $e_2$ outgoing from it there exists an automorphism $\psi$ of $G$ such that $\psi(v) = v$, $\psi(e_1) = e_2$.

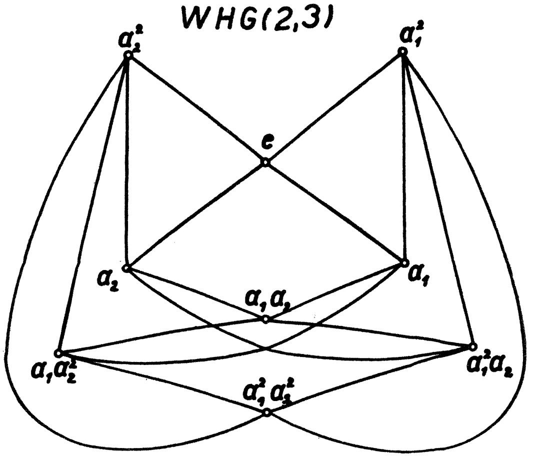


Fig. 1.

($\gamma'$) For any vertex $v$ of $G$ and any two edges $h_1$, $h_2$ incoming into it there exists an automorphism $\psi'$ of $G$ such that $\psi'(v) = v$, $\psi'(h_1) = h_2$.

A digraph satisfying ($\alpha$) will be called a symmetric digraph.

Let $v$ be a vertex of a symmetric digraph $G$. Let $w_1, \ldots, w_k$ be the terminal vertices of the edges outgoing from $v$ and let $\varphi_i$ for each $i$ be some automorphism of $G$ such that $\varphi_i(v) = w_i$. Let $\mathfrak{G}$ be the subgroup of the group of all automorphisms of $G$ which is generated by the mappings $\varphi_i$. We shall express a theorem analogous to Theorem 1.

**Theorem 7.** *Let $G_0$ be the subgraph of $G$ generated by all vertices $\psi(v)$, where $\psi \in \mathfrak{G}$. Then $G_0$ is a connected component of $G$.*

Proof is analogous to that of Theorem 1.

The proof of the following theorem is analogous to that of Theorem 2.

**Theorem 8.** *Let the group of all automorphisms of the digraph $G$ be Abelian. Let $\psi_1(v) = \psi_2(v)$ for some elements $\psi_1$, $\psi_2$ of this group. Then $\psi_1 \equiv \psi_2$.*

Analogously to the definition of $G(\mathfrak{G}, A)$ we shall define the digraph $\vec{G}(\mathfrak{G}, A)$. Let $\mathfrak{G}$ be some group, $A$ its subset. The vertices of $\vec{G}(\mathfrak{G}, A)$ are elements of $\mathfrak{G}$ and there exists an edge going from $x$ into $y$ in $\vec{G}(\mathfrak{G}, A)$ if and only if $x^{-1}y$ is in $A$.

Theorems 7 and 8 show that any symmetric digraph is a homomorphic image (even isomorphic in the case when $\mathfrak{G}$ is Abelian) of some $\vec{G}(\mathfrak{G}, A)$. It is connected if and only if $A$ is a system of generators of $\mathfrak{G}$.

The proofs of the following theorems are analogous to the proofs of analogous theorems for undirected graphs.

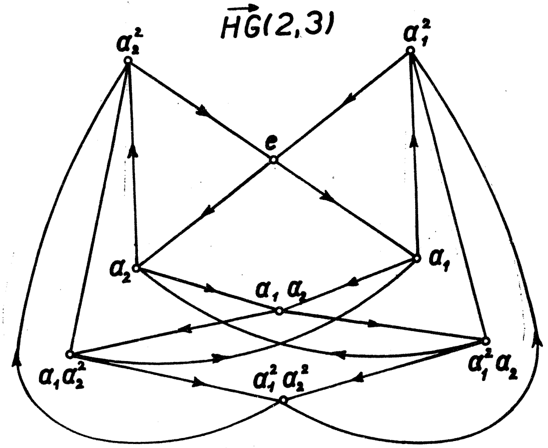


Fig. 2.

**Theorem 9.** *The digraph $\vec{G}(\mathfrak{G}, A)$ is symmetric.*

**Theorem 10.** *Let $\varphi$ be an automorphism of the group $\mathfrak{G}$ such that $\varphi(A) = A$. Then $\varphi$ is also an automorphism of $\vec{G}(\mathfrak{G}, A)$.*

**Theorem 11.** *Let $\mathfrak{G}$ be a group, $A$ system of its generators. Let any permutation of $A$ be induced by some automorphism of $\mathfrak{G}$. Then $\vec{G}(\mathfrak{G}, A)$ is a connected homogeneous digraph.*

**Theorem 12.** *Let $\mathfrak{G}$ be a group, $A$ a system of its generators. To any two elements $a_1, a_2$ of $A$ let there exist an automorphism $\varphi$ of $\mathfrak{G}$ such that $\varphi(A) = A$, $\varphi(a_1) = a_2$. Then $\vec{G}(\mathfrak{G}, A)$ is a connected weakly homogeneous digraph.*

We see that the formulations of Theorems 11 and 12 are simpler than the formulations of the corresponding Theorems 5 and 6. In Theorem 11 we use $A$ instead of $A^*$, because of the simpler definition of $\vec{G}(\mathfrak{G}, A)$ (compared with $G(\mathfrak{G}, A)$). In Theorem 12 we do not assume that $\mathfrak{G}$ is Abelian, because we need not consider the mappings of elements of $A$ onto inverses of elements of $A$.

Any finitely generated group satisfying the assumptions of Theorem 11 can be evidently expressed as a factor-group of the free group with $k$ generators, derived from it by the system of defining relations symmetric with respect to all free generators.

Analogously to the graphs $WHG(k, r)$ we shall construct the digraphs $\overrightarrow{HG}(k, r)$. They are not only weakly homogeneous, but even homogeneous. If $\mathfrak{G}$ is the direct product of cyclic groups $\mathfrak{A}_1, \ldots, \mathfrak{A}_k$ which have all the same order, we see that any permutation of the generators $a_1, \ldots, a_k$ of the groups $\mathfrak{A}_1, \ldots, \mathfrak{A}_k$ is induced by some automorphism of $\mathfrak{G}$ and this automorphism preserves $A = \{a_1, \ldots, a_k\}$. The digraph

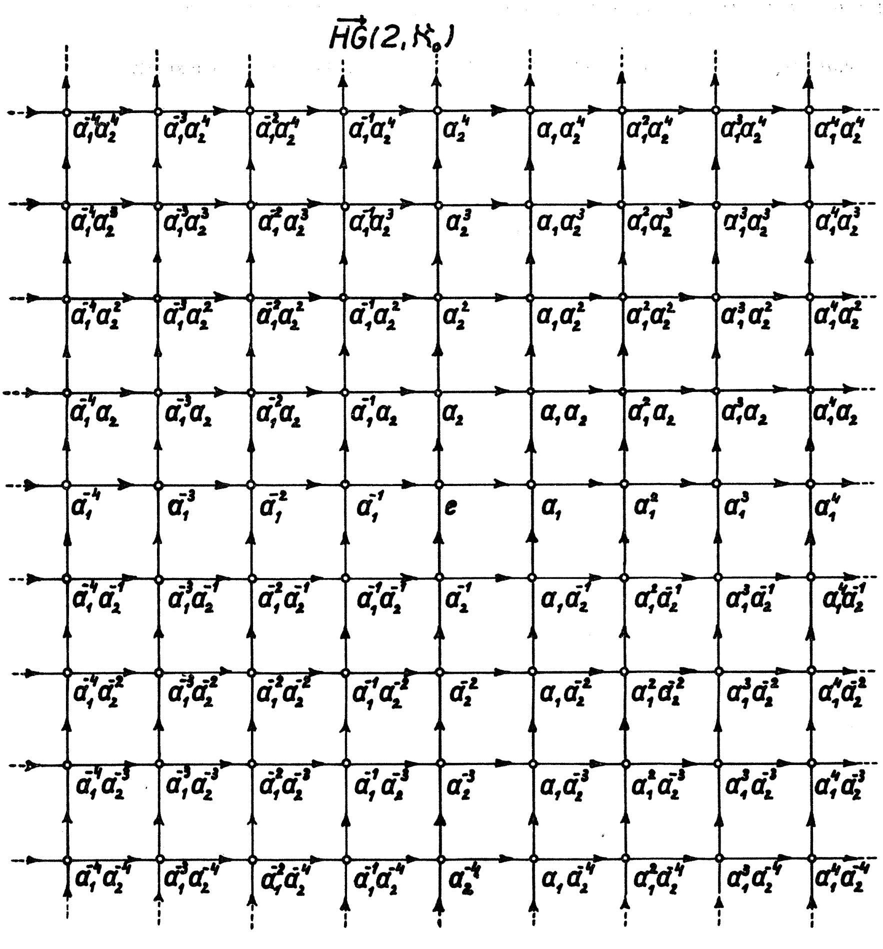


Fig. 3.

$\overrightarrow{HG}(1, r)$ is the directed circuit of the length $r$. The digraph $\overrightarrow{HG}(k, 2)$ is created from the graph $Q_k$ of the $k$-dimensional cube by substituting any undirected edge by the pair of directed edges opposite to each other. The digraph $\overrightarrow{HG}(2, 3)$ is on Fig. 2. As in the undirected case, we can take for $k$ also any cardinal number, for $r$ a cardinal number less than or equal to $\aleph_0$. The digraph $\overrightarrow{HG}(2, \aleph_0)$ is on Fig. 3.

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