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Robert L. Hemminger

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ISOMORPHISM-INDUCED LINE ISOMORPHISMS ON PSEUDOGRAPHS*)

ROBERT L. HEMMINGER, Nashville

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WHITNEY [3] has showed that for finite graphs a line isomorphism is induced by an isomorphism except in the cases illustrated by Figures 1.1 through 1.4. JUNG [2] has since showed that the assumption of finiteness in Whitney's result can be dropped. In this paper we characterize the line isomorphisms on pseudographs that are induced by isomorphisms. This generalization is obtained by a method similar to that used by Jung in the infinite graph case.

1. Line isomorphisms induced by isomorphisms. Throughout the paper G and H will denote connected pseudographs and σ will be a one-to-one function of the lines of G onto the lines of H. However, the results hold for disconnected pseudographs (without isolated points) as well since line isomorphisms and isomorphisms preserve connected components. A number of definitions follow: any terms used and not defined in this paper can be found in HARARY [1].

Definitions. (a) A pseudograph G is *an ordered triple* $(V(G), E(G), \Gamma(G))$ where $V(G) \neq \emptyset$ (it may be an infinite set) and $\Gamma(G) : E(G) \to V(G) \& V(G)$ (unordered pairs). The elements of $V(G)$ are called *points* and the elements of $E(G)$ are called *lines*. If $\alpha \in E(G)$ and $\Gamma(G)(\alpha) = (a, a)$, α is called *a loop*; if $\alpha, \beta \in E(G)$ with $\Gamma(G)(\alpha) = \Gamma(G)(\beta)$, then α and β are called *multiple lines* and α is called *a multiple of* β.

(b) A multigraph is a *pseudograph* that has no loops. A graph is a pseudograph with neither loops nor multiple lines. In a graph one usually identifies a line α with $\Gamma(\alpha)$.

(c) A function $\tau : V(G) \to V(H)$ is *an isomorphism of G onto H* if τ is one-to-one, onto, and for every pair of points, a and a', ($a = a'$ is allowed) the cardinal number

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of lines between a and a' is the same as the cardinal number of lines between $\tau(a)$ and $\tau(a')$.

(d) A function $\sigma : E(G) \to E(H)$ is *a line isomorphism of G onto H* if σ is one-to-one, onto, and for each distinct pair of lines α and β, α and β are adjacent if and only if $\sigma(\alpha)$ and $\tau(\beta)$ are adjacent.

(e) A line isomorphism σ is induced by an isomorphism τ if for each $\alpha \in E(G)$ we have $\Gamma(G)(\alpha) = (a, a')$ if and only if $\Gamma(H)(\sigma(\alpha)) = (\tau(a), \tau(a'))$.

(f) For $a \in V(G)$ we let $S(a) = \{\alpha \in E(G) : \Gamma(G)(\alpha) = (a, a') \text{ for some } a' \in V(G)\}$. $S(a)$ is called *the cluster at a* and C is called *a cluster* if $C = S(a)$ for at least one $a \in V(G)$. C is called *a star*, or *star with center a*, if $C \subseteq S(a)$.

(g) We say that a line function $\sigma : E(G) \to E(H)$ preserves a particular type of line set if $\sigma(A)$ is of that type whenever A is. In particular, we will refer to line functions that preserve loops, multiple lines, stars, and clusters.

(h) A point a of G is *a terminal point* of G if there is a point $a' \neq a$ with $S(a) \subseteq S(a')$.

(i) If $A \subseteq V(G)$, then the subpseudograph of G generated by A is the one whose point set is A and whose line set consists of all lines of G that are incident only with points in A. If $A \subseteq E(G)$, then the subpseudograph of G generated by A is the one whose line set is A and whose point set consists of those incident with some line of A. In either case we use $\bar{A}$ to denote the subpseudograph generated by A.

Theorem 1. *Let G and H be connected pseudographs and let $\sigma : E(G) \to E(H)$ be one-to-one and onto. Then the following three conditions are equivalent*:

(1) *σ is induced by an isomorphism,*
(2) *σ and σ^{-1} are line isomorphisms that preserve loops, multiple lines, and stars, and*
(3) *σ and σ^{-1} are cluster preserving and $E(G)$ and $E(H)$ do not each consist of a set of multiple lines; non-loops for one and loops for the other.*

Proof. (1) implies (2). Immediate from the definitions and the observation that τ^{-1} induces σ^{-1} if τ induces σ.

(2) implies (3). Let σ be as in (2) and let $a \in V(G)$. Then $\sigma(S(a)) \subseteq S(b)$ for at least one $b \in V(H)$.

If $\sigma(S(a)) \subseteq S(b) \cap S(b')$ with $b' \neq b$ then $\sigma(S(a))$ is a set of non-loop, multiple lines. Hence $S(a) = \sigma^{-1}(\sigma(S(a))$ is also a set of non-loop, multiple lines, so a is a terminal point of G. Let a' be the point adjacent to a. Without loss of generality, we assume that $\sigma(S(a')) \subseteq S(b')$. But $\sigma^{-1}(S(b)) \subseteq S(a)$ or $\sigma^{-1}(S(b)) \subseteq S(a')$. If the former, then $\sigma(S(a)) = S(b)$ directly. If the latter, then $S(b) \subseteq \sigma(S(a')) \subseteq S(b')$, so $\sigma(S(a)) = S(b) \cap S(b') = S(b)$. Thus $\sigma(S(a))$ is a cluster.

If $\sigma(S(a)) \subseteq S(b)$ and $\sigma(S(a)) \nsubseteq S(c)$ for $c \neq b$, then neither $S(a)$ nor $\sigma(S(a))$ is a set of multiple lines unless $S(a)$ is a set of loops. In either case $\sigma^{-1}(S(b)) \subseteq S(a)$ and so $\sigma(S(a)) = S(b)$.

By symmetry, σ^{-1} is also cluster preserving and obviously $E(G)$ consists of a set of loops if and only if $E(H)$ does.

(3) implies (1). Let σ be as in (3). Because of the exclusion given in (3) we can assume that G and H have at least three points and that $S(a) = S(a')$ if and only if $a = a'$, where a and a' are in either G or H. Thus the function $\sigma^* : V(G) \to V(H)$ defined by the equation $\sigma(S(a)) = S(\sigma^*(a))$ is both well-defined and one-to-one. Moreover it is onto since $\sigma(\sigma^{-1}(S(b))) = S(b)$. For $a \neq a'$ in G we have, since σ is one-to-one, $\sigma(S(a) \cap S(a')) = \sigma(S(a)) \cap \sigma(S(a')) = S(\sigma^*(a)) \cap S(\sigma^*(a'))$; i.e., there are the same cardinal number of lines between a and a' as there are between $\sigma^*(a)$ and $\sigma^*(a')$. And, since α is a loop in G if and only if $\alpha \in S(a)$ for exactly one a in G, σ and σ^{-1} take loops to loops. So there are the same cardinal number of loops at a as at $\sigma^*(a)$. Thus σ is an isomorphism of G onto H. Clearly σ^* induces σ since $\sigma(S(a)) = S(\sigma^*(a))$ (this takes care of loops) and for $a \neq a'$, $\sigma(S(a) \cap S(a')) = S(\sigma^*(a)) \cap S(\sigma^*(a'))$ so that $\Gamma(\alpha) = (a, a')$ if and only if $\Gamma(\sigma(\alpha)) = (\sigma^*(a), \sigma^*(a'))$. That completes the proof of the theorem.

Corollary (Jung [2]): *Let G and H be connected graphs and let σ be a line isomorphism of G onto H. Then σ is induced by an isomorphism if and only if σ is different than the line isomorphisms illustrated in Figures* 1.1 *through* 1.4.

Proof. We use condition (2) of the theorem. It is clear that every line isomorphism preserves k-stars where $k \neq 3$. If $S(a)$ is a 3-star and $\sigma(S(a))$ is not, then $\sigma(S(a))$ is the line set of a triangle. Thus G has no line adjacent to just one line of $S(a)$. Hence G has either 3, 4, 5, or 6 lines and the four possibilities obviously correspond to the four cases illustrated in Figures 1.1 through 1.4.

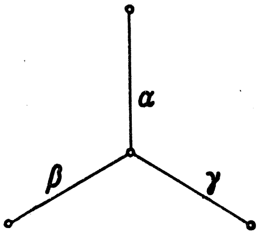
2. Line isomorphisms not induced by isomorphisms. Before describing such line isomorphisms we give a few more definitions.

Definitions. (a) We say that G' is *a multiversion of* G if G is a subpseudograph of G', with the same point set as G', such that there is a partition $\{M(\alpha) : \alpha \in E(G)\}$ of $E(G')$ with $M(\alpha)$ a set of multiple lines containing α for each $\alpha \in E(G)$.

(b) We say that a pair of pseudographs, G' and H', is *a σ-multiversion of the pair G and H* if G' and H' are multiversions of G and H with line partitions $\{M(\alpha) : \alpha \in E(G)\}$ and $\{N(\beta) : \beta \in E(H)\}$ respectively, such that there is a one to one onto function $\sigma' : E(G') \to E(H')$ with $\sigma'(M(\alpha)) = N(\sigma(\alpha))$ for each $\alpha \in E(G)$. We note that σ' is a line isomorphism if and only if σ is and that σ' is induced by an isomorphism if and only if σ is.

(c) We say that G_1 is *a terminal piece of* G based at a if $G = G_1 \cup G_2$ with $V(G_1) \cap V(G_2) = \{a\}$ (Note that a need not be a cutpoint of G since G_1 (or G_2) may only contain loops at a). We will be interested in three types of terminal pieces: (1) A terminal star based at its center, (2) a terminal multitriangle, and (3) a terminal cluster, with center b, that is based at a with $a \neq b$, with $S(b) \nsubseteq S(a)$, and with $|S(b) \cap S(a)| \geqq 2$. Hereafter we will refer to the latter as a non-center-based terminal cluster. These three types of terminal pieces are illustrated in Figure 2.2.

We are now ready to describe pseudographs G and H that are appropriate generalization of the graphs in Figures 1.1 through 1.4.



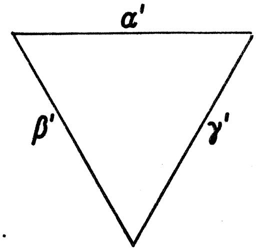
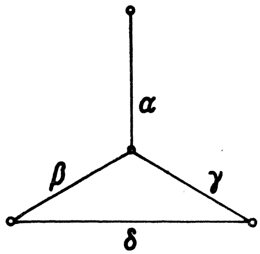


Fig. 1.1.



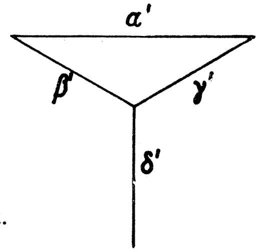
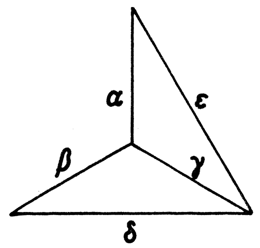


Fig. 1.2.



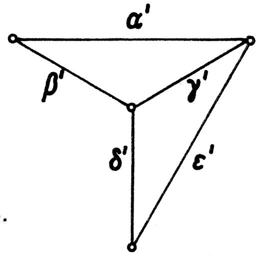
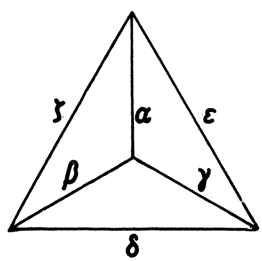


Fig. 1.3.



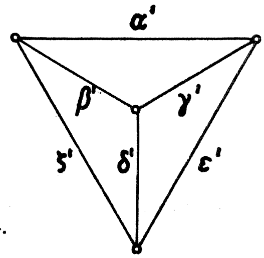


Fig. 1.4.

(1) G and H are pseudographs with the property that their line sets consist of pairwise adjacent lines and σ is arbitrary except that σ or σ^{-1} fails to preserve stars, loops, or multiple lines. Thus G and H are clusters or multitriangles. In contrast to the case for graphs, the line isomorphism might not be induced by an isomorphism even though G and H are both clusters, or are both multitriangles. Also there is no restriction on the cardinality of $E(G)$ even when G is a cluster and H is a multitriangle. We should further point out that line isomorphic clusters and line isomorphic multitriangles need not be isomorphic.

(2) The pair, G and H, is a τ-multiversion of the pair of graphs in Figures 1.3 or 1.4 where $\tau(\chi) = \chi'$ for each line χ in either figure.

(3) Let the pair, G' and H', be a τ-multiversion (with partitions $\{M(\chi)\}$ and $\{(N(\chi')\}$ respectively) of the pair of graphs in Figure 1.2 where $\tau(\chi) = \chi'$ for each line. Then $M(\alpha)$ is a center-based terminal star of G'. Let a be the base point. Similarly let b be the base point of the center-based terminal star $N(\delta')$ of H'. Let G be obtained from G' by replacing $M(\alpha)$ by a terminal star A that is center-based at a, with $|A| = |M(\alpha)|$, and that is otherwise arbitrary. Similarly H is obtained from H' by replacing $N(\delta')$ by a terminal star D that is center-based at b and has $|D| = |N(\delta')|$. Let σ be any one-to-one function from $E(G)$ onto $E(H)$ with the property that $\sigma(A) = N(\alpha')$, $\sigma(M(\beta) \cup M(\gamma)) = N(\beta') \cup N(\gamma')$, and $\sigma(M(\delta)) = D$. Then σ is a line isomorphism of G onto H that is not induced by an isomorphism. This situation is illustrated in Figure 2.1 where the dotted lines are used to indicate that σ must map the lines of G in a region created by the dotted lines onto the lines of H in that region.

Lemma. *If σ and σ^{-1} preserve stars and G and H have non-terminal points then there is a one-to-one function σ^* from the non-terminal points of G onto the non-terminal points of H such that $\sigma(S(a)) = S(\sigma^*(a))$ for each non-terminal point a of G. Moreover, if we let $S'(a)$ be the maximal terminal star of G that is center-based at a where a is a non-terminal point of G, then $\sigma(S'(a)) = S'(\sigma^*(a))$.*

Proof. For $a \in V(G)$, $\sigma(S(a)) \subseteq S(b)$ for some b and $S(a) \subseteq \sigma^{-1}(S(b)) \subseteq S(a')$ for some $a' \in V(G)$. If a is non-terminal then $a' = a$ so $\sigma(S(a)) = S(b)$. If $\sigma(S(a)) = S(b')$ also, with $b' \neq b$, then $S(b) = S(b')$ and H is a star with only terminal points. Since this is excluded by hypothesis σ induces a function σ^*, from non-terminal points of G to points of H, which is defined by the equation $\sigma(S(a)) = S(\sigma^*(a))$.

Now σ^* is one to one for if a and a' are distinct non-terminal points of G then $S(\sigma^*(a)) = \sigma(S(a)) \neq \sigma(S(a')) = S(\sigma^*(a'))$, and hence $\sigma^*(a) \neq \sigma^*(a')$. If $\sigma^*(a)$ is a terminal point then $S(\sigma^*(a) \subseteq S(b')$ for some $b' \neq \sigma^*(a)$. But then, as before, $S(\sigma^*(a)) = S(b')$ contrary to hypothesis. Thus σ^* takes non-terminal points to non-terminal points and since the same arguments hold relative to σ^{-1} we conclude that σ^* is onto the set of non-terminal points of H. That establishes the first part of the Lemma.

If a and a' are non-terminal points then $\sigma(S(a) \cap S(a')) = \sigma(S(a)) \cap \sigma(S(a')) = S(\sigma^*(a)) \cap S(\sigma^*(a'))$, i.e., σ preserves multiple lines between non-terminal points and it takes lines between non-terminal points to such lines. Similarly for σ^{-1}.

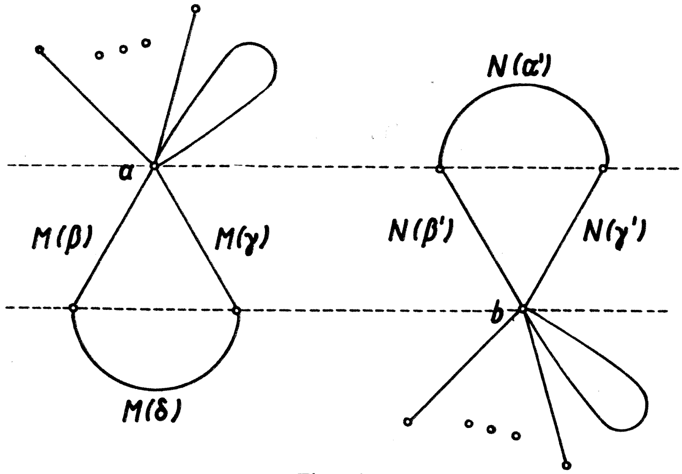


Fig. 2.1.

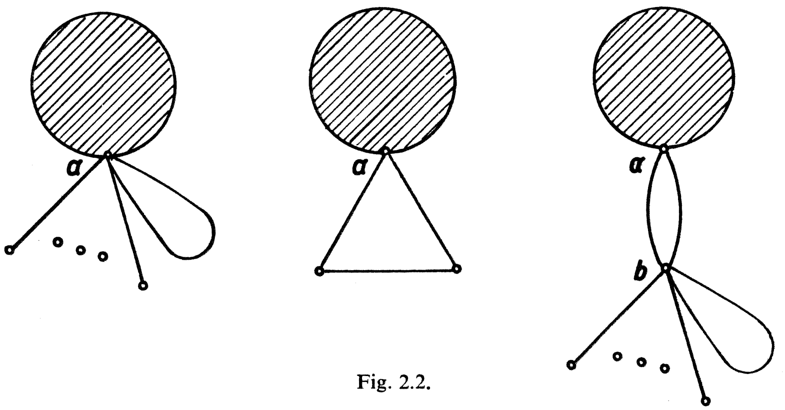


Fig. 2.2.

If α is a loop of G at a then a is a nonterminal point so $\sigma(\alpha) \in S(\sigma^*(a))$. Moreover $\sigma(\alpha) \in S'(\sigma^*(a))$ for otherwise we would have $\alpha = \sigma^{-1}(\sigma(\alpha)) \in S(a) - S'(a)$.

Combining the results of the last two paragraphs we conclude that $\sigma(S'(a)) \subseteq S'(\sigma^*(a))$. But equality must hold since we likewise have $\sigma^{-1}(S'(\sigma^*(a))) \subseteq S'(a)$.

Theorem 2. *Let σ be a line isomorphism of G onto H, G and H connected pseudographs, where σ is not induced by an isomorphism and where G and H are not as in* (1), (2), *or* (3) *above. Then G has a terminal piece G_1 based at a and H has a terminal* *piece* H_1 *based at* b *such that* $\sigma(E(G_1)) = E(H_1)$, $\sigma(S(a)) = S(b)$, σ *restricted to* G_1 *is not induced by an isomorphism, and where* G_1 *and* H_1 *are one of the following*: (a) *center-based terminal stars,* (b) *terminal triangles, or* (c) *a terminal triangle and a non-center-based terminal cluster.*

Proof. By Theorem 1, σ or σ^{-1} fails to preserve loops, multiple lines, or stars. Suppose σ fails to preserve stars, i.e., there is an $a \in V(G)$ such that $\overline{\sigma(S(a))}$ is a multitriangle. Let $A = \overline{S(a)}$ and $G_1 = \overline{V(A)}$.

We will assume a is not a terminal point for if it is then $S(a) \subseteq S(a')$ for some non-terminal point a' and $\sigma(S(a'))$ is not a star. Note that G_1 includes all loops at points of A.

Suppose that, for each $a' \neq a$, G_1 is not a terminal piece of G at a'. Then there are distinct non-loops α and β that are not in G_1 and that are incident with different points of G_1. But then $\sigma(\alpha)$ and $\sigma(\beta)$ are both incident with the multitriangle $\overline{\sigma(S(a))}$. Hence α and β are both adjacent to some $\gamma \in S(a)$. Under our assumptions this means that $\alpha \in E(G_1)$ or $\beta \in E(G_1)$ which is a contradiction. We conclude that G_1 is a terminal piece of G at a' for some $a' \neq a$.

We consider two cases: $G_1 \neq G$ and $G_1 = G$. First suppose that $G_1 \neq G$. Then there is an $\alpha \in S(a') - E(G_1)$. Since a is not a terminal point, $\sigma(\alpha)$ is incident with the multitriangle $\overline{\sigma(S(a))}$ at a unique point, say b'. Let b and b'' be the other two points of $\overline{\sigma(S(a))}$. Then $\sigma(S(a) \cap S(a')) \subseteq (S(b') \cap S(b'')) \cup (S(b') \cap S(b))$. On the other hand if $\sigma(\beta)$ is in the latter set and $\beta \notin S(a) \cap S(a')$ then G_1 must be a multitriangle on $\{a, a', a''\}$ where $\Gamma(\beta) = (a', a'')$. In either case $\sigma(S(a')) = S(b')$ and $\sigma(S(a') \cap E(G_1)) = S(b') \cap E(\overline{\{b, b', b''\}})$. Also in either case $\sigma(S(a) \cap S(a'))$ has non-empty intersection with both $S(b') \cap S(b)$ and $S(b') \cap S(b'')$. Because of this distribution of $S(a') \cap E(G_1)$ by σ we conclude, in either case, that $H_1 = \overline{\{b, b', b''\}}$ is a terminal piece of H at b' and we get the conclusion of the theorem with G_1 and H_1 as in (b) or (c) where loops at a' and b' have been removed from G_1 and H_1.

Suppose next that $G_1 = G$. If G_1 has a loop α at $a' \neq a$ then $\sigma(\alpha)$ is incident with a unique point b' of $\overline{\sigma(S(a))}$. If we let $G_1' = G_1 - \{\alpha\}$ and $H_1 = H - \{\sigma(\alpha)\}$ then as in the last paragraph we get the desired conclusion with G_1' and H_1 as in (b) or (c). Thus $E(G_1) - S(a)$ contains no loops. Let $\alpha \in E(G_1) - S(a)$ (possible since G is not a star), say $\Gamma(\alpha) = (a', a'')$. Then $\sigma(\alpha) \in S(b)$ where b is one of the points of $\overline{\sigma(S(a))}$ and $\sigma(\alpha)$ is not a multiple of any line in $\sigma(S(a))$ for then G and H would be multitriangles. Moreover if α and β are multiple lines then $\sigma(\beta) \in S(b)$ and is not a multiple of any line in $\sigma(S(a))$. If $E(G_1)$ consists of $S(a)$ and the multiples of α then we conclude that G and H are as in (3) preceding the theorem. The existence of one more such set of multiple lines in G easily implies that the pair, G and H, is a τ-multiversion of the pair of graphs in Figure 1.3. The existence of two more such sets of multiple lines in G implies that the pair, G and H, are τ-multiversions of the pair of graphs in Figure 1.4.

There are no other possibilities in the case that σ fails to preserve stars and by the results in that case we see that the same conclusions will hold for σ if we assume that σ^{-1} fails to preserve stars.

Thus we assume that σ and σ^{-1} preserve stars. Since G and H are not stars we can apply the Lemma. From it we get center-based terminal stars $G_1 = S'(a)$ and $H_1 = = S'(\sigma^*(a))$ as in the conclusion of the theorem.

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Author's address: Vanderbilt University, Nashville, Tennessee 37203, U.S.A.