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GREEN'S RELATIONS ON A COMPACT SEMIGROUP

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Let \( S \) be a semigroup. Then \( \overset{\circ}{K} \) will denote the equivalence on \( S \): for \( a, b \in S \), \( a \overset{\circ}{K} b \) if and only if there exist positive integers \( m, n \) such that \( a^m = b^n \). In [1] J. T. Sedlock studies necessary and sufficient conditions on a periodic semigroup \( S \) in order that \( \overset{\circ}{K} \) coincide with any one of the Green relations [2]. In our paper [3] we considered an arbitrary semigroup having similar properties.

The fact that any element \( x \) of a compact semigroup \( S \) belongs to some idempotent (see [4]) leads us to define an equivalence \( \overset{\circ}{K}_T \) on \( S \) by: for \( a, b \in S \), \( a \overset{\circ}{K}_T b \) if and only if the elements \( a, b \) belong to the same idempotent. The purpose of this article is to investigate the structure of compact semigroups such that \( \overset{\circ}{K}_T \) coincides with any one of the Green relations.

Let \( \mathcal{C}(S) \) denote the set of all \( \overset{\circ}{C} \)-closure operations for a non-empty set \( S \), i.e.

\[
\begin{align*}
(0) & \quad U \in \mathcal{C}(S) \iff U : \exp S \rightarrow \exp S \\
(1) & \quad U(\emptyset) = \emptyset , \\
(2) & \quad A \subseteq B \subseteq S \Rightarrow U(A) \subseteq U(B) , \\
(3) & \quad A \subseteq U(A) \quad \text{for each} \quad A \subseteq S , \\
(4) & \quad U(U(A)) = U(A) \quad \text{for each} \quad A \subseteq S
\end{align*}
\]

hold.

A subset \( A \) of \( S \) will be called \( U \)-closed if \( U(A) = A \). The set of all \( U \)-closed subsets of \( S \) will be denoted by \( \mathcal{F}(U) \).

Let \( U, V \in \mathcal{C}(S) \). Then we define

\[
U \leq V \iff U(A) \subseteq V(A) \quad \text{for each} \quad A \subseteq S .
\]
We have

\begin{align*}
    (5) \quad \mathcal{F}(U \lor V) &= \mathcal{F}(U) \cap \mathcal{F}(V), \\
    (6) \quad U \leq V &\iff \mathcal{F}(V) \subseteq \mathcal{F}(U).
\end{align*}

We shall denote by \(\mathcal{A}(S)\) the set of all \(\mathcal{A}\)-closure operations for a set \(S\), i.e. \(\mathcal{A}(S) \subseteq \mathcal{G}(S)\) and for every \(U \in \mathcal{A}(S)\) and for every \(A \subseteq S\)

\begin{align*}
    (7) \quad U(A) &= \bigcup_{x \in A} U(x)
\end{align*}

holds. If \(U, V \in \mathcal{A}(S)\) then

\begin{align*}
    (8) \quad U \leq V &\iff U(x) \subseteq V(x) \quad \text{for each} \quad x \in S.
\end{align*}

Let \(U \in \mathcal{G}(S)\). We define \(U^* \in \mathcal{A}(S)\). If \(A \subseteq S\) then \(x \in U^*(A)\) if and only if \(U(x) \cap \cap A \neq \emptyset\). For \(U, V \in \mathcal{G}(S)\) we have

\begin{align*}
    (9) \quad U \leq V &\Rightarrow U^* \leq V^*, \\
    (10) \quad U(x) &= U^*(x) \quad \text{for every} \quad x \in S, \\
    (11) \quad U^* &= U^{**} \quad \text{and} \quad U^{**} \leq U.
\end{align*}

See [5].

Let \(U \in \mathcal{G}(S)\). We shall introduce the equivalence \(\circ U\) on \(S\) by: for \(x, y \in S\), \(x \circ U y\) if and only if \(U(x) = U(y)\). For any element \(x\) of \(S\), let \(U_x\) denote the \(\circ U\)-class of \(S\) containing \(x\). If \(U, V \in \mathcal{G}(S)\) then we have

\begin{align*}
    (12) \quad U \leq V &\Rightarrow \circ U \subseteq \circ V, \\
    (13) \quad \circ(U \land V) &= \circ U \land \circ V, \\
    (14) \quad x \circ U y &\iff x \in U(y) \quad \text{and} \quad y \in U(x).
\end{align*}

See [3].

Let \(S\) be an arbitrary semigroup. For any \(A \subseteq S\), \(A \neq \emptyset\), let us put \(L(A) = SA \cup A\) and \(R(A) = AS \cup A\). Finally, \(L(\emptyset) = R(\emptyset) = \emptyset\). Clearly \(L, R \in \mathcal{A}(S)\). Put \(M = L \lor R\), \(H = L \land R\), \(F(L) = [F(R), F(M), F(H)]\) is the set of all left [right, two-sided, quasi] ideals of \(S\) (including \(\emptyset\)). It is known that

\begin{align*}
    (15) \quad H_e \text{ is the maximal subgroup of } S \text{ belonging to the idempotent } e.
\end{align*}

Put \(P(\emptyset) = \emptyset\). If \(A \subseteq S\), \(A \neq \emptyset\), then by \(P(A)\) we denote the subsemigroup generated by all elements of \(A\). Evidently \(P \in \mathcal{G}(S)\), \(P \leq H\) and \(F(P)\) is the set of all subsemigroups of \(S\) (including \(\emptyset\)). See [5].

Let \(K = P^* \lor P^{**}\). Then \(K = K^*\) and \(x^* K y\) if and only if there exist positive integers \(n, m\) such that \(x^n = y^m\). See [3].
Let now $S$ be a compact (Hausdorff) semigroup. If $A \subset S$, then by $T(A)$ we denote the closure of $A$. It is known that $T \in \mathcal{C}(S)$ and

$$T(A \cup B) = T(A) \cup T(B) \text{ for } A \subset S \text{ and } B \subset S,$$

$$T(x) = \{x\} \text{ for each } x \in S.$$

We shall prove that

$$T(AB) = T(A) T(B) \text{ for } \emptyset \neq A \subset S \text{ and for } \emptyset \neq B \subset S.$$

Actually, it follows from 2.1.3 [6] that $T(A) T(B) \subset T(AB)$. Since $T(A)$, $T(B)$ are compact, it follows from 2.1.5 [6] that $T(A) T(B)$ is also compact and thus we have $T(AB) \subset T(A) T(B)$. By (3), we obtain that $A \subset T(A)$, $B \subset T(B)$. Hence $AB \subset T(A) T(B)$. Using (2), we have $T(AB) \subset T(T(A) T(B)) = T(A) T(B)$. This means that (18) holds.

Put $P_T = P \cup T$. It follows from (5) that $\mathcal{F}(P_T)$ is the set of all closed subsemigroups of $S$ (including $\emptyset$). It is known from [4] that for $x \in S$

$$P_T(x) = T(P(x)) \text{ is the commutative subsemigroup having a unique idempotent.}$$

Lemma 1. Let $A \subset S$. Then $A \in \mathcal{F}(P_T)$ if and only if

$$P_T(x) \cap A \neq \emptyset \Rightarrow x \in A$$

for every $x \in S$.

Proof. Let $A \in \mathcal{F}(P_T)$. If $P_T(x) \cap A \neq \emptyset$ for some $x \in S$, then there exists $y$ such that $y \in P_T(x)$ and $y \in A$. It follows from (2) that $x \in P_T^*)(y) = P_T^*(A) = A$.

Let (20) hold for every $x \in S$. Evidently $P_T^*(A) = \mathcal{F}(P_T^*)$. If $A \neq \emptyset$, then by (7) we have $P_T^*(A) = \bigcup_{x \in A} P_T^*(x)$. If $y \in P_T^*(A)$, then $y \in P_T^*(x)$ for some $x \in A$. Since $x \in P_T(y)$, it follows by (20) that $y \in A$. Therefore, $P_T^*(A) \subset A$. It follows from (3) that $A = P_T^*(A) \in \mathcal{F}(P_T^*)$.

Remark. Let $A \subset S$, $A \neq \emptyset$. Then $P_T^*(A)$ is the set of all almost nilpotent elements (in topological sense) with respect to $A$. (See [7].)

Proof. If $x \in P_T^*(A)$, then there exists $y \in A \cap P_T(x) = A \cap T(P(x))$. If $O$ is an arbitrary neighbourhood of $A$, then $O$ is also a neighbourhood of $y$ and thus $x^n \in O$ for some positive integer $n$. Therefore, the element $x$ is almost nilpotent with respect to $A$.

If $x$ is an almost nilpotent element with respect to $A$, then in every neighbourhood of $A$ there exists at least one element of $P(x)$. Suppose that $x \notin P_T^*(A)$. This implies that $P_T(x) \cap A = \emptyset$. Evidently $O = S - P_T(x) = S - T(P(x))$ is a neighbourhood of $A$ and $A \cap P(x) = \emptyset$, which is a contradiction. Therefore, $x \in P_T^*(A)$. 

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Definition. \( K_T = P_T^* \lor P_T^{**} \).

Lemma 2. \( K_T = K_T^* \).

Proof. (9) implies that \( P_T^{**} \leq K_T^* \) and \( P_T^{***} \leq K_T^* \). It follows from (11) that \( K_T = P_T^* \lor P_T^{**} \leq K_T^* \). According to (9) and (11), we have \( K_T^* \leq K_T^{***} \leq K_T \). Hence \( K_T = K_T^* \).

Lemma 3. Let \( x, e \in S \) and let \( e^2 = e \). If \( e \in P_T(x) \), then \( x^e K_T e \).

Proof. If \( e \in P_T(x) \), then \( x \in P_T^*(e) \subset K_T(e) \). It follows from (10) that \( e \in P_T(x) = P_T^{**}(x) \subset K_T(x) \). (14) implies that \( x^e K_T e \).

Lemma 4. Let \( e, f \in S \) and let \( e^2 = e, f^2 = f \). If \( e^e K_T f \), then \( e = f \).

Proof. Using (14) we obtain \( e \in K_T(f) \). Let \( A = \{ u \in S | f \in P_T(u) \} \). We shall show that \( A \in \mathcal{F}(K_T) = \mathcal{F}(P_T^* \lor P_T^{**}) = \mathcal{F}(P_T^*) \cap \mathcal{F}(P_T^{**}) \) (see (5)). If \( P_T(x) \cap A \neq \emptyset \) for some \( x \in S \), then there exists \( u \) such that \( u \in A \) and \( u \in P_T(x) \). This implies that \( f \in P_T(u) \subset P_T(x) \) and thus we have \( x \in A \). By Lemma 1, \( A \in \mathcal{F}(P_T^*) \). If \( x \in P_T^{**}(A) \), then by (7) and (10) we have \( x \in P_T^{***(u)} = P_T(u) \) for some \( u \in A \). This implies that \( P_T(x) \subset P_T(u) \). Since \( f \in P_T(u) \), hence, by (19), \( f \in P_T(x) \) and thus \( x \in A \). This means that \( P_T^{**}(A) \subset A \) and according to (3) we obtain \( A = P_T^{**}(A) \in \mathcal{F}(P_T^{**}) \). Therefore, \( A \in \mathcal{F}(K_T) \). Since \( f \in A \), (2) and (4) imply \( e \in K_T(f) \subset A \) and thus we have \( f \in P_T(e) = = \{ e \} \) (see (17)). Therefore, \( e = f \).

Theorem 1. Let \( x, y \in S \). Then \( x^e K_T y \) if and only if there exists an idempotent \( e \) of \( S \) such that

\[
(21) \quad e \in P_T(x) \cap P_T(y).
\]

Proof. Let \( x^e K_T y \). By (19) there exist \( e, f \) of \( S \) such that \( e = e^2 \in P_T(x) \) and \( f = f^2 \in P_T(y) \). Lemma 3 implies that \( e^e K_T f \). According to Lemma 4, we have \( f = e \) and \( e \in P_T(x) \cap P_T(y) \).

Let (21) hold. Then according to Lemma 3, we have \( x^e K_T e \) and \( y^e K_T e \). This implies that \( x^e K_T y \).

Lemma 5. \( K \subseteq K_T \) and \( e^e K \subseteq e^e K_T \).

Proof. Evidently \( P \subseteq P \lor T = P_T \) and (9) implies that \( P^* \leq P_T^* \) and \( P^{**} \leq P_T^{**} \). Therefore, \( K \subseteq K_T \). By (12), we have \( e^e K \subseteq e^e K_T \).

Lemma 6. If \( e \) is an idempotent of \( S \), then \( eK_T e = K_T e = H_e \).

Proof. See Theorem 8 in [4].
Put $L_T = L \lor T$, $R_T = R \lor T$ and $M_T = M \lor T$. Note that $M_T = L_T \lor R_T$. It follows from (5) that $\mathcal{F}(L_T)[\mathcal{F}(R_T), \mathcal{F}(M_T)]$ is the set of all closed left [right, two-sided] deals of $S$ (including $0$).

**Lemma 7.** We have

1. $^{o}L = ^{o}L_T \land L = L_T^{**}$,
2. $^{o}R = ^{o}R_T \land R = R_T^{**}$,
3. $^{o}M = ^{o}M_T \land M = M_T^{**}$.

**Proof.** Let $x \in S$. It follows from (16), (17), (18) and (5) that $L(x) \in \mathcal{F}(L_T)$. This implies that $L(x) = L_T(x)$. By (14), we have $^{o}L = ^{o}L_T$. Further, by (10), we obtain that $L(x) = L_T^{**}(x)$. According to (8), we have $L = L_T^{**}$.

Analogously we can prove the statements 2 and 3.

**Lemma 8.** $L \leq R$ if and only if $L_T \leq R_T$.

**Proof.** If $L \leq R$, then $L_T = L \lor T \leq R \lor T = R_T$. Let $L_T \leq R_T$. Then $L_T(x) \subseteq R_T(x)$ for every $x \in S$. According to the proof of Lemma 7, we have $L(x) \subseteq R(x)$. It follows from (8) that $L \leq R$.

**Lemma 9.** If $A \subseteq S$, $A \neq \emptyset$, then

$$\bigcap_{x \in A} xS = \bigcap_{x \in T(A)} xS$$

holds.

**Proof.** Let $z \in \bigcap_{x \in A} xS$. Suppose that $z \notin \bigcap_{x \in T(A)} xS$. It follows that $z \notin uS$ for some $u \in T(A)$. By (17) and (18), $uS$ is a closed subset of the compact semigroup $S$ and there exists a neighbourhood $O$ of $uS$ such that $z \notin O$. Evidently $uS \in O$ for every $a \in S$. It follows from the continuity of multiplication that there exist neighbourhoods $O_a(u)$ of $u$ and $O(a)$ of $a$ such that $O_a(u) \cap O(a) = O$. It is clear that $S = \bigcup_{a \in S} O(a)$.

Since $S$ is a compact semigroup, there exists a finite system $O(a_1), O(a_2), \ldots, O(a_n)$ which also covers $S$. If we put $O_0(u) = O_{a_1}(u) \cap O_{a_2}(u) \cap \ldots \cap O_{a_n}(u)$, then $O_0(u) S \subseteq O$. Since $O_0(u)$ is a neighbourhood of $u$, there exists $x \in A \cap O_0(u)$. Evidently $z \in xS$. If $z = xb$ for some $b \in S$, then $z \in O_0(u) S \subseteq O$ which is a contradiction. Hence $z \in \bigcap_{x \in T(A)} xS$. According to (3), we have $A \subseteq T(A)$ so that $\bigcap_{x \in T(A)} xS \subseteq xS$. Hence (22) holds.

**Lemma 10.** If $A \subseteq S$, $A \neq \emptyset$, then

$$\bigcap_{x \in A} SxS = \bigcap_{x \in T(A)} SxS$$

holds.
Proof. Let \( z \in SxS \). Suppose that \( z \notin SxS \). It follows that \( z \notin SuS \) for some \( u \in T(A) \). By (17) and (18), \( SuS \) is a closed subset of the compact semigroup \( S \) and there exists a neighbourhood \( O \) of \( SuS \) such that \( z \notin O \). Evidently \( aub \in O \) for every \( a, b \in S \). It follows from the proof of Lemma 9 that there exist neighbourhoods \( O'_a(au) \) of \( au \) such that \( O'_a(au) S \subset O \) for every \( a \in S \). The continuity of multiplication implies that there exist neighbourhoods \( O(a) \) of \( a \) and \( O'_a(u) \) of \( u \) such that \( O(a) O'_a(u) \subset O'_a(au) \). Evidently \( S = \bigcup_{a \in S} O(a) \). Since \( S \) is a compact semigroup, there exists a finite system \( O(a_1), O(a_2), \ldots, O(a_n) \) which also covers \( S \). If we put \( O_0(u) = O_{a_1}(u) \cap O_{a_2}(u) \cap \ldots \cap O_{a_n}(u) \), then \( SO_0(u) S \subset \bigcap_{i=1}^n O'_a(a_iu) S \subset O \). Since \( O_0(u) \) is a neighbourhood of \( u \), there exists \( x \in S \). Evidently \( z \in SxS \). If \( z = axb \) for some \( a, b \in S \), then \( z \notin O_0(u) S \) which is a contradiction. Hence \( z \notin \bigcap_{x \in T(A)} SxS \). The rest of the proof is analogous to that of Lemma 9.

**Theorem 2.** The following conditions on a semigroup \( S \) are equivalent:

1. \( S \) is right regular;
2. \( P^*_T \leq R_T \);
3. \( K_T \leq R_T \);
4. \( \circ K_T \subset \circ R \).

Proof. 1 \( \Rightarrow \) 2. Let \( S \) be a right regular semigroup. Let \( A \) be a closed right ideal of \( S \), i.e. \( A \in \mathcal{F}(R_T) \). If \( u \in P_T(x) \cap A \), then by (2) we have \( R_T(u) \subset A \). Since \( S \) is right regular, \( x \in x^nS \) for every positive integer \( n \). It follows from Lemma 9 that \( x \in \bigcap_{x \in S} vS = \bigcap_{x \in P_T(x)} vS \). This implies that \( x \in uS \subset R_T(u) \subset A \). By Lemma 1 we have \( A \in \mathcal{F}(P^*_T) \). It follows from (6) that \( P^*_T \leq R_T \).

2 \( \Rightarrow \) 3. Suppose \( P^*_T \leq R_T \). Since \( P \leq R \), it holds \( P_T \leq R_T \). According to (9) and Lemma 7, we have \( P^*_T \leq R^*_T = R \leq R_T \). Thus \( K_T = P^*_T \vee P^*_T \leq R_T \).

3 \( \Rightarrow \) 4. This follows from (12) and from Lemma 7.

4 \( \Rightarrow \) 1. If \( \circ K_T \subset \circ R \), then by Lemma 5 we have \( \circ K \subset K_T \subset \circ R \). It follows from Theorem 6 in [3] that \( S \) is right regular.

The dual statement reads as follows:

**Theorem 3.** The following conditions on a semigroup \( S \) are equivalent:

1. \( S \) is left regular;
2. \( P^*_T \leq L_T \);
3. \( K_T \leq L_T \);
4. \( \circ K_T \subset \circ L \).
**Theorem 4.** The following conditions on a semigroup $S$ are equivalent:

1. $S$ is a union of groups;
2. $P_T^* \leq R_T \land L_T$;
3. $K_T \leq R_T \land L_T$;
4. $^\circ K_T \leq ^\circ H$.

**Proof.** $1 \Rightarrow 2 \Rightarrow 3$. This follows from Theorem 2 and Theorem 3.

$3 \Rightarrow 4$. It follows from (12), Lemma 7 and (13) that $^\circ K_T \subseteq ^\circ (R_T \land L_T) = ^\circ R_T \land \cap ^\circ L_T = ^\circ R \land ^\circ L = ^\circ (R \land L) = ^\circ H$.

$4 \Rightarrow 1$. If $^\circ K_T \subseteq ^\circ H$, then by Lemma 5 we have $^\circ K \subseteq ^\circ H$. It follows from Theorem 8 in [3] that $S$ is a union of groups.

**Theorem 5.** The following conditions on a semigroup $S$ are equivalent:

1. $S$ is intraregular;
2. $P_T^* \leq M_T$;
3. $K_T \leq M_T$;
4. $^\circ K_T \leq ^\circ M$.

**Proof.** $1 \Rightarrow 2$. Let $S$ be an intraregular semigroup. Let $A$ be a closed two-sided ideal of $S$, i.e. $A \in \mathcal{F}(M_T)$. If $u \in P_T(x) \land A \ (x \in S)$, then by (2) we have $M_T(u) \subseteq A$. For every positive integer $n$, we have $x^{n+2} \in Sx^nS$. It follows from Theorem 9 of [3] and (6) that $Sx^nS \in \mathcal{F}(M_T) \subseteq \mathcal{F}(P_T^*)$. Lemma 2 in [3] implies that $x \in Sx^nS$. It follows from Lemma 10 that $x \in \bigcap_{e \in P_T(x)} SuS = \bigcap_{e \in P_T(x)} SvS$. This implies that $x \in Sx^eS \subseteq M_T(u) \subseteq A$. It follows from Lemma 1 that $A \in \mathcal{F}(P_T^*)$. By (6) we have $P_T^* \leq M_T$.

$2 \Rightarrow 3 \Rightarrow 4$. The proof is analogous to the proof of Theorem 2.

$4 \Rightarrow 1$. If $^\circ K_T \subseteq ^\circ M$, then by Lemma 5 we have $^\circ K \subseteq ^\circ M$. It follows from Theorem 9 of [3] that $S$ is intraregular.

**Theorem 6.** The conditions of Theorems 2, 3, 4 and 5 and the following condition on a semigroup $S$ are equivalent:

$$^\circ K_T = ^\circ H.$$ 

**Proof.** 2 of Theorem 2 $\Rightarrow$ 2 of Theorem 5. If $P_T^* \leq R_T$, then $P_T^* \leq R_T \leq M_T$.

2 of Theorem 5 $\Rightarrow$ 1 of Theorem 4. Let $x \in S$. It follows from (19) that $e \in P_T(x)$ where $e^2 = e$. By Theorem 1 and Lemma 6 we have $ex \in H_e$. (15) and (6) imply that $e \in SexS \in \mathcal{F}(M_T) \subseteq \mathcal{F}(P_T^*)$. According to Lemma 1, we obtain that $x \in SexS$. Then there exist $a, b \in S$ such that $x = axb$. If we put $c = ae$, then $x = cxb$ and $c = ce$. This implies that $x = c^nxb^n$ and $c^n = c^ne$ for any positive integer $n$. Let
f \in P_\tau(c) where f^2 = f (see (19)). Then by Lemma 9 we have \( x \in \bigcap_{v \in P(c)} vS = \bigcap_{v \in P_\tau(c)} vS \) so that \( x \in fS \). Since \( P(c) = P(c) e \), we obtain by (18) and (17) that \( P_\tau(c) = P_\tau(c) e \). Since \( f \in P_\tau(c) e \), it holds \( f = ue \) for some \( u \in P_\tau(c) \). Therefore \( f = u e^2 = fe \). Since \( x \in fS \), \( x = fz \) holds for some \( z \in S \). This implies that \( x = fz = f^2z = fx = fex \). According to (19), we have \( x = xe \) and thus \( R(xe) = R(x) = R(fex) = R(fxe) = R(xe) \). Therefore \( R(xe) = R(ex) = eR(x) \). Since \( x \in eR(x) \), it follows that \( x = ew \) for some \( w \in R(x) \). This implies that \( x = ew = e^2w = ex \in H_e \). Hence \( S \) is a union of groups.

4 of Theorem 4 \( \Rightarrow \)^ 4K_t = \(^0H \). Suppose \( \^0K_t \subset \^0H \). If \( \^0K_t \neq \^0H \), then there exist \( x, y \in S \) such that \( K_{tx} \neq K_{ty} \) and \( K_{tx} \subset H_x = H_y \supset K_{ty} \). Let \( e \in P_\tau(x) (e^2 = e) \) and let \( f \in P_\tau(y) (f^2 = f) \). Lemma 3 implies that \( e \in K_{tx} \) and \( f \in K_{ty} \) and thus we obtain that \( e, f \in H_x \). According to (15), we have \( e = f \) so that \( K_{tx} = K_{te} = K_{ty} \) which is a contradiction. Hence \( \^0K_t = \^0H \).

\( \^0K_t = \^0H \Rightarrow 4 \) of Theorem 3. This follows from \( \^0H \subset \^0L \) (see (12)).

2 of Theorem 3 \( \Rightarrow 1 \) of Theorem 2. Let \( x \in S \). It follows from (19) that \( e \in P_\tau(x) \) where \( e^2 = e \). Since \( e \in Se \in \mathcal{F}(L_T) \subset \mathcal{F}(P_\tau^\bullet) \) (see (6)), hence \( P_\tau(x) \cap Se \neq 0 \). By Lemma 1 we have that \( x \in Se \). Therefore \( x = ue \) for some \( u \in S \) and so \( x = ue = ue^2 = xe \). According to Lemma 6 and Lemma 3, we have \( x \in H_e \). This implies that \( S \) is a union of groups and therefore, \( S \) is right regular.

**Theorem 7.** The following conditions on a semigroup \( S \) are equivalent:

1. \( S \) is a semilattice of right groups;
2. \( S \) is a union of groups and \( L_T \leq R_T \);
3. \( P_\tau^\bullet \leq L_T \leq R_T \);
4. \( K_T \leq L_T \leq R_T \);
5. \( ^0K_T \subset ^0L \subset ^0R \);
6. \( ^0K_T = ^0L \).

**Proof.** 1 \( \Rightarrow 2 \). It follows from Theorem 10 of [3] that \( S \) is a union of groups and \( L \leq R \). By Lemma 8 we have \( L_T \leq R_T \).

2 \( \Rightarrow 4 \Rightarrow 5 \). This follows from Theorem 3, Theorem 4 and from (12).

5 \( \Rightarrow 6 \). If \( ^0L \subset ^0R \), then by Theorem 6 and (13) we have \( ^0K_T = ^0H = ^0L \).

6 \( \Rightarrow 1 \). If \( ^0K_T = ^0L \), then by Theorem 6 and Lemma 5 we have \( ^0K \subset ^0K_T = ^0L = ^0H \subset ^0R \). Theorem 10 in [3] implies that \( S \) is a semilattice of right groups.

We have:

**Theorem 8.** The following conditions on a semigroup \( S \) are equivalent:

1. \( S \) is a semilattice of left groups;
2. \( S \) is a union of groups and \( R_T \leq L_T \);

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Theorem 9. The following conditions on a semigroup $S$ are equivalent:

1. $S$ is a semilattice of groups;
2. $S$ is a union of groups and $L_T = R_T$;
3. $P^*_T \leq R_T \leq L_T$;
4. $K_T \leq R_T \leq L_T$;
5. $^\circ K_T \in ^\circ R \in ^\circ L$;
6. $^\circ K_T = ^\circ R$.

Proof. $1 \Rightarrow 2$. It follows from Theorem 12 of [3] that $S$ is a union of groups and $L = R$. Thus we have $L_T = R_T$.

$2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 6$. This follows from Theorem 7 and Theorem 8.

$6 \Rightarrow 7$. It follows from Theorems 7 and 8 that $L_T = R_T$. According to Lemma 8 and its dual, we have $L = R = M$ so that $^\circ K_T = ^\circ L = ^\circ M$.

$7 \Rightarrow 1$. Theorem 6 implies that $^\circ H = ^\circ K_T = ^\circ M = ^\circ L = ^\circ R$. According to Lemma 5, we have $^\circ K \in ^\circ L = ^\circ R$. It follows from Theorem 12 in [3] that $S$ is a semilattice of groups.

References


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