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*Czechoslovak Mathematical Journal*, Vol. 22 (1972), No. 1, 176–179

Persistent URL: http://dml.cz/dmlcz/101084

Czechoslovak Mathematical Journal, 22 (97) 1972, Praha

# CORRECTION AND ADDITION TO MY PAPER "THE NORMAL FORM AND THE STABILITY OF SOLUTIONS OF A SYSTEM OF DIFFERENTIAL EQUATIONS IN THE COMPLEX DOMAIN"

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(Received April 29, 1971)

**1. Correction.** The correct form of the condition $(Q_4)$ in Theorem 3,4 of [1] (pp. 53–59) is the following.

Given an arbitrary $p \in \mathscr{P}(\lambda)$, there exists such a complex number $\alpha_p$ that

$$\{\eta_k\}_p = \alpha_p \lambda_k \quad (k = 1, 2, \ldots, l)\,. \tag{3,16}$$

Notation 3,4 is now unnecessary and the proof of the theorem is to be modified in an obvious manner: $\nu = 0$ and hence $\mathscr{Q}_{k,\nu}(\lambda) = \mathscr{P}_k(\lambda)$, $\mathscr{S}_{k,\nu}(\lambda) = \mathsf{E}[p \in \mathscr{P}_k(\lambda) : : p_k \geqq 0] = \tilde{\mathscr{P}}_k(\lambda) = \tilde{\mathscr{P}}(\lambda)$ $(k = 1, 2, \ldots, l)$, $\mathscr{M}'_\nu = \emptyset$, $\mathscr{S}'_\nu(\lambda) = \emptyset$ and $\beta(y) \equiv 0$. Instead of (3,20) we have

$$\{g_k\}_p [(p, \lambda) - \lambda_k] + \{Y_k\}_p =$$

$$= \{\tilde{X}_k(g)\}_p + \sum_{j=l+2}^{n} \varepsilon_j(p_j + 1)\{g_k\}_{\tilde{p}(j)} - \sum_{\substack{\omega+\sigma=p \\ \omega \in \mathscr{M}_3 \\ \sigma \in \tilde{\mathscr{P}}(\lambda)}} \left(\sum_{j=1}^{l} p_j \lambda_j\right) \alpha_\sigma \{g_k\}_\omega -$$

$$- \sum_{\substack{\omega+\sigma=p \\ \omega,\sigma \in \mathscr{M}_2}} \left(\sum_{j=l+1}^{n} (\omega_j + 1)\{g_k\}_{\hat{\omega}(j)}\{Y_j\}_\sigma\right) \quad \text{for} \quad p \in \mathscr{M}_2\,, \quad k = 1, 1, \ldots, l\,,$$

$$\{Y_k\}_p = \{\tilde{X}_k(g)\}_p \quad \text{for} \quad p \in \mathscr{M}_2\,, \quad k = l + 1, l + 2, \ldots, n\,.$$

Under the original assumption $(Q_4)$ the implication (3,20) ⇒ (3,21) is false. (The author is indebted to A. D. BRJUNO who discovered this error.) Even by Theorem 2 of A. D. Brjuno from [2] divergence can occur in the case.

Theorem 3,4 is now a special case of Theorem 1 from [2]. (The remark at the beginning of sec. 3,4 in [1] concerns only [3], not [2].) Corollary (p. 59) is no more a direct consequence of Theorem 3,4, but it can be proved in a quite similar way as Theorem 3,4. (Under assumptions of this Corollary the relation (3,19') holds also for $j = 1, 2, \ldots, l$ and the implication (3,20) $\Rightarrow$ (3,21) in the original form is true.)

Finally let us note that the proofs of all results of A. D. Brjuno will be given in [4] and [5].

**2. Addition.** The following simple generalization of the well-known Cartan's Uniqueness Theorem is in a close connection with Theorem 4,2 A from [1] (pp. 66–67). The proof of Theorem 4,2 A could be based on it and on the method of L. Reich from [6] and [7].

In the following we make use of notations and conventions from [8], in particular of those introduced in chapters I–III.

Proposition. *Let $D$ be a bounded domain in the space $C_n$ of $n$ complex variables and let $\varrho_1, \varrho_2, \ldots, \varrho_n$ be such complex numbers that*

$$1 = |\varrho_1| = |\varrho_2| = \ldots = |\varrho_m| > |\varrho_{m+1}| \geqq \ldots \geqq |\varrho_n| > 0 .$$

*Then the mapping $T$*

$$x_j' = \varrho_j x_j + [\textit{higher powers}] \quad (j = 1, 2, \ldots, n) \tag{1}$$

*is formally similar to the mapping*

$$\begin{aligned} y_j' &= \varrho_j y_j & (j = 1, 2, \ldots, m)\,, \\ y_j' &= \varrho_j y_j + [\textit{higher powers}] & (j = m+1, m+2, \ldots, n)\,, \end{aligned} \tag{2}$$

*whenever $T$ maps $D$ into $D$.*

Proof. By [6] any mapping (1) is formally similar to a mapping of the form

$$y_j = \varrho_j y_j + \sum_{|p| \geqq 2} \{V_j\}_p \, y^p = \varrho_j y_j + \sum_{r \geqq 2} \mathscr{V}_{j,r}(y) \quad (j = 1, 2, \ldots, n)\,, \tag{3}$$

where $p = (p_1, p_2, \ldots, p_n)$, $|p| = p_1 + p_2 + \ldots + p_n$, $y^p = y_1^{p_1} y_2^{p_2} \ldots y_n^{p_n}$, $\mathscr{V}_{j,r}(y)$ is a polynomial consisting of all terms in (3) of the order $r$ and $\{V_j\}_p = 0$ whenever $1 \leqq j \leqq m$ and $\varrho^p \neq \varrho_j$. (Certainly if $1 \leqq j \leqq m$ and $|p_{m+1}| + |p_{m+2}| + \ldots + |p_n| > 0$, then $\{V_j\}_p = 0$.)

Let us order the coefficients $\{V_j\}_p$ ($|p| \geqq 2$, $j = 1, 2, \ldots, m$) in the usual way. ($\{V_j\}_p \prec \{V_k\}_q$ iff the first nonzero number in the set $\{|q| - |p|, k - j, q_1 - p_1, \ldots, q_n - p_n\}$ is positive.) Let $\{V_k\}_q$ be the first unvanishing coefficient. Then there is a polynomial transformation $U$ ($z_j' = z_j + \tilde{U}_j(z)$, $j = 1, 2, \ldots, n$, where $\tilde{U}_j$ are finite polynomials) such that $V = U^{-1}TU$ has the form

$$y_j' = \varrho_j y_j + [\textit{powers of degree higher than } |\boldsymbol{q}|] \qquad (j = 1, 2, \ldots, k-1),$$

$$y_k' = \varrho_k y_k + \{V_k\}_{\boldsymbol{q}}\, y^{\boldsymbol{q}} + \mathscr{W}_{k,\boldsymbol{q}}(y) + [\textit{higher powers}],$$

$$y_j = \varrho_j y_j + [\textit{powers of degree higher than } |\boldsymbol{q}| - 1] \qquad (j = k+1, k+2, \ldots, m),$$

$$y_j = \varrho_j y_j + [\textit{higher powers}] \qquad (j = m+1, m+2, \ldots, n).$$

($\mathscr{W}_{k,\boldsymbol{q}}(y)$ is a polynomial of the variables $y_1, y_2, \ldots, y_m$ which contains terms of degree $|\boldsymbol{q}|$ and not preceding $\{V_k\}_{\boldsymbol{q}}$ in the given ordering.)

Let $s$ be an arbitrary natural number and let the mapping $\boldsymbol{T}^s$ be given by

$$y_j^{(s)} = \sum_{|\boldsymbol{p}| \geqq 1} \{T_j^s\}_{\boldsymbol{p}}\, y^{\boldsymbol{p}} \quad (j = 1, 2, \ldots, n)$$

$(\boldsymbol{T}^2(y) = \boldsymbol{T}(\boldsymbol{T}(y)),\ \boldsymbol{T}^s(y) = \boldsymbol{T}(\boldsymbol{T}^{s-1}(y)))$.

Let $\boldsymbol{T}(\boldsymbol{D}) \subset \boldsymbol{D}$. Then given an arbitrary $\boldsymbol{p}$ with $|\boldsymbol{p}| \geqq 1$, there exists a real number $C_{\boldsymbol{p}}$ such that

$$|\{T_j^s\}_{\boldsymbol{p}}| \leqq C_{\boldsymbol{p}} \quad (j = 1, 2, \ldots, n;\ s = 1, 2, \ldots),$$

i.e. $\{\boldsymbol{T}^s\}$ $(s = 1, 2, \ldots)$ is weakly bounded. By [8] (I, §3, p. 12) $\{\boldsymbol{V}^s\} = \{\boldsymbol{U}^{-1}\boldsymbol{T}^s\boldsymbol{U}\}$ $(s = 1, 2, \ldots)$ is weakly bounded, too. But in $\boldsymbol{V}^2$

$$y_j^{(2)} = \varrho_j^2 y_j + [\textit{powers of degree higher than } |\boldsymbol{q}|] \quad (j = 1, 2, \ldots, k-1),$$

$$y_k^{(2)} = \varrho_k^2 y_k + \varrho_k \{V_k\}_{\boldsymbol{q}}\, y^{\boldsymbol{q}} + \varrho_k \mathscr{W}_{k,\boldsymbol{q}}(y) + \{V_k\}_{\boldsymbol{q}}\, \varrho^{\boldsymbol{q}} y^{\boldsymbol{q}} + \mathscr{W}_{k,\boldsymbol{q}}(\varrho_1 y_1, \ldots, \varrho_m y_m) +$$
$$+ [\textit{higher powers}] = \varrho_k^2 y_k + 2\varrho_k \{V_k\}_{\boldsymbol{q}}\, y^{\boldsymbol{q}} + \mathscr{W}_{k,\boldsymbol{q}}^{(2)}(y) + [\textit{higher powers}],$$

where $\mathscr{W}_{k,\boldsymbol{q}}^{(2)}$ contains only terms of degree $|\boldsymbol{q}|$ and not preceding $\{V_k\}_{\boldsymbol{q}}$. Generally in $\boldsymbol{V}^s$

$$y_k^{(s)} = \varrho_k^s y_k + s\varrho_k^{s-1}\{V_k\}_{\boldsymbol{q}}\, y^{\boldsymbol{q}} + \varrho_k^{s-1}\mathscr{W}_{k,\boldsymbol{q}}(y) + \varrho_k^{s-2}\mathscr{W}_{k,\boldsymbol{q}}(\varrho_1 y_1, \ldots, \varrho_m y_m) + \ldots$$
$$\ldots + \mathscr{W}_{k,\boldsymbol{q}}(\varrho_1^{s-1} y_1, \ldots, \varrho_m^{s-1} y_m) + [\textit{higher powers}] =$$
$$= \varrho_k^s y_k + s\varrho_k^{s-1}\{V_k\}_{\boldsymbol{q}}\, y^{\boldsymbol{q}} + \mathscr{W}_{k,\boldsymbol{q}}^{(s)}(y) + [\textit{higher powers}],$$

where $\mathscr{W}_{k,\boldsymbol{q}}^{(s)}$ contains only terms of degree $|\boldsymbol{q}|$ and not preceding $\{V_k\}_{\boldsymbol{q}}$. It is clear that the set $\{s\varrho_k^{s-1}\{V_k\}_{\boldsymbol{q}}\}$ $(s = 1, 2, \ldots)$ is bounded iff $\{V_k\}_{\boldsymbol{q}} = 0$.

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