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CONCERNING ALMOST REALCOMPACTIFICATIONS

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INTRODUCTION

It is known that every completely regular Hausdorff space has a realcompactification νX , contained in the Stone-Čech compactification βX , with the property that every continuous function from X to a realcompact space can be extended to νX . In [1], FROLÍK has defined and investigated almost realcompactness. The purpose of this paper is to show that every Hausdorff space has an almost realcompactification ρX , which is contained in the Katětov H -closed extension $\varkappa X$, which is a projective maximum in the class of almost realcompactifications of X , and which has an extension property similar to (although necessarily weaker than) that of νX .

1. PRELIMINARIES

1.1. Definitions. An *open filter* is a non-empty collection of open sets \mathcal{U} such that

- (1) $\emptyset \notin \mathcal{U}$, and
- (2) if $U, V \in \mathcal{U}$ and $G = \text{int}(G) \supset U \cap V$, then $G \in \mathcal{U}$.

An open filter \mathcal{U} is said to have the *countable closure intersection property* (abbreviated c.c.i.p.) provided that for each countable subcollection $C \subset \mathcal{U}$, $\bigcap \{\text{cl } U \mid U \in C\} \neq \emptyset$. An *open ultrafilter* is an open filter which is maximal in the collection of open filters.

1.2. Definitions. Let X be a Hausdorff space. X is said to be

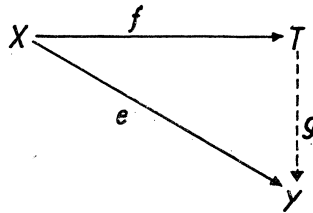
- (1) *H-closed* provided that every open ultrafilter on X converges; or, equivalently, X is closed in every Hausdorff space in which it is embedded.
- (2) *almost realcompact* provided that every open ultrafilter on X with c.c.i.p. converges.

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All H -closed, all Lindelöf and all realcompact spaces are almost realcompact. However, (unlike the situation for α -spaces [7]) there are spaces which are not almost realcompact; e.g. the space of all ordinals less than the first uncountable ordinal.

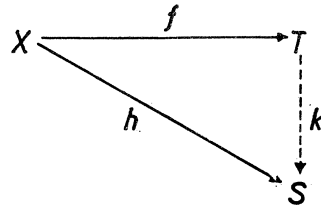
1.3. Definitions. An *almost realcompactification* (resp. *H -closed extension*) of a Hausdorff space X is a pair (Y, e) where Y is an almost realcompact space (resp. H -closed space) and $e : X \rightarrow Y$ is a dense embedding.

If $f : X \rightarrow T$ is a dense embedding, then (T, f) is called a q -*extension* of X provided that for any almost realcompactification (Y, e) of X , there exists a continuous function $g : T \rightarrow Y$ such that $g \cdot f = e$, i.e.,



commutes. If, in addition, T is almost realcompact, then (T, f) is called a *projective maximum* in the class of almost realcompactifications of X .

Notice that by the definition, a projective maximum (T, f) in the class of almost realcompactifications of X is essentially unique, i.e., if (S, h) is any other projective maximum, then there exists a homeomorphism $k : T \rightarrow S$ such that



commutes.

In [8] KATĚTOV has constructed an H -closed extension $(\varkappa X, e)$ for each Hausdorff space X , which in [5] has been shown to be a projective maximum in the class of H -closed extensions of X . We briefly indicate the construction of $\varkappa X$. Let X^\vee be the set of all non-convergent open ultrafilters on X . Let $\varkappa X = X \cup X^\vee$ topologized as follows: a base for the neighborhoods of a point p in $\varkappa X$ is its neighborhood system in X if $p \in X$, and is $\{\{p\} \cup G \mid G = \text{int } G \in p\}$ if $p \in X^\vee$. $e : X \rightarrow \varkappa X$ is the inclusion function (which is open).

1.4. Lemma. *If X is a dense subset of Y and \mathcal{U}' is an open ultrafilter on Y , then $\mathcal{U} = \{U \cap X \mid U \in \mathcal{U}'\}$ is an open ultrafilter on X which converges in Y if and only if \mathcal{U}' converges in Y .*

1.5. Lemma. *Suppose that X is dense in Y and \mathcal{U} is an open ultrafilter on X . Then $\mathcal{U}' = \{G \mid G \text{ is open in } Y \text{ and } G \cap X \in \mathcal{U}\}$ is an open ultrafilter on Y which converges in Y if and only if \mathcal{U} converges in Y .*

1.6. Corollary. *If X is an open, dense subset of Y , then \mathcal{U} and \mathcal{U}' , as described in the above lemmas, are related by: $\mathcal{U} = \{U \in \mathcal{U}' \mid U \subset X\}$.*

2. AN ALMOST REALCOMPACTIFICATION OF X

2.1. Definition. For any Hausdorff space X , X^c will be the set of all non-convergent open ultrafilters on X with c.c.i.p., and ϱX will be $X \cup X^c$, considered as a subspace of $\varkappa X$.

2.2. Lemma. *Suppose that each of T and Y contains X as a dense subset, where Y is almost realcompact. Then every continuous $g : T \rightarrow Y$ whose restriction on X is the inclusion map, can be extended to a continuous mapping from ϱT into Y .*

Proof. Let $\mathcal{P} \in \varrho T - T = T$. Then \mathcal{P} is a non-convergent open ultrafilter on T with c.c.i.p. Let $\mathcal{U} = \{P \cap X \mid P \in \mathcal{P}\}$ and $\mathcal{G} = \{G \text{ open in } Y \mid G \cap X \in \mathcal{U}\}$. By Lemmas 1.4. and 1.5, \mathcal{G} is an open ultrafilter on Y . We wish to show that \mathcal{G} has c.c.i.p. in Y .

Suppose there exists a countable collection $G_n \in \mathcal{G}$ such that $\bigcap_n \text{cl}_Y G_n = \emptyset$. Let $g^{-1}[G_n] = P_n$. Since $P_n \cap X = G_n \cap X \in \mathcal{U}$, by the maximality of \mathcal{P} and Lemma 1.5, we have that $P_n \in \mathcal{P}$. Now, $g[P_n] \subset G_n$ implies that $g[\text{cl}_T P_n] \subset \text{cl}_Y g[P_n] \subset \text{cl}_Y G_n$. Therefore $\bigcap_n g[\text{cl}_T P_n] = \emptyset$, and consequently $\bigcap_n \text{cl}_T P_n = \emptyset$. But this contradicts the fact that \mathcal{P} has c.c.i.p. in T . Since Y is almost realcompact, there exists some point $p \in Y$ such that \mathcal{G} converges to p in Y .

Let us define $f(\mathcal{P}) = p$ for $\mathcal{P} \in \varrho T - T$ and $f(t) = g(t)$ for $t \in T$.

It is clear that f is continuous at each $t \in T$ because T is open in ϱT . Consider $\mathcal{P} \in \varrho T - T$, where $f(\mathcal{P}) = p$ as above. Let W be an open neighborhood of p in Y . Since $W \in \mathcal{G}$, it follows that $G = g^{-1}[W] \in \mathcal{P}$. Thus $G \cup \{\mathcal{P}\}$ is an open neighborhood of \mathcal{P} in ϱT such that $f[G \cup \{\mathcal{P}\}] = g[G] \cup \{p\} \subset W$.

2.3. Theorem. *Let $e : X \rightarrow \varrho X$ be the inclusion map. Then $(\varrho X, e)$ is an almost realcompactification of X which is a projective maximum in the class of almost realcompactifications of X .*

Proof. Since X is open and dense in $\varkappa X$, it is clear that it is open and dense in ϱX . Let \mathcal{U} be an open ultrafilter on ϱX with c.c.i.p. in ϱX . We wish to show that \mathcal{U} converges in ϱX . Let $\mathcal{P} = \{U \subset X \mid U \in \mathcal{U}\}$. By Lemma 1.4 and Corollary 1.6, \mathcal{P} is an open ultrafilter on X , whose convergence in X implies the convergence of \mathcal{U} in ϱX . Suppose that \mathcal{P} does not converge in X . We will show that it then belongs to X^c . Let $U_n \in \mathcal{P}$, $n = 1, 2, \dots$. Clearly, by the construction of ϱX ,

$$\text{cl}_{\varrho X} U_n = \text{cl}_X U_n \cup A_n \quad \text{where} \quad A_n = \{\gamma \in X^c \mid U_n \in \gamma\}.$$

Notice that when $m \neq n$, $\text{cl}_X U_m \cap A_n = \emptyset$. Thus since \mathcal{U} has c.c.i.p. in ϱX , we have

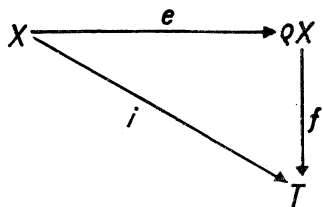
$$\emptyset \neq \bigcap_n (\text{cl}_X U_n \cup A_n) = \left(\bigcap_n \text{cl}_X U_n \right) \cup \left(\bigcap_n A_n \right).$$

If $\bigcap_n A_n = \emptyset$, then $\bigcap_n \text{cl}_X U_n \neq \emptyset$. If $\hat{\mathcal{P}} \in \bigcap_n A_n$, then each $U_n \in \hat{\mathcal{P}}$. Since $\hat{\mathcal{P}}$ has c.c.i.p. in X , it follows that $\bigcap_n \text{cl}_X U_n \neq \emptyset$. Consequently \mathcal{P} has c.c.i.p. in X , i.e., $\mathcal{P} \in X^c$. Now every open neighborhood of \mathcal{P} in ϱX contains a member of \mathcal{P} (which is contained in \mathcal{U}). Thus \mathcal{U} converges to \mathcal{P} . To show that $(\varrho X, e)$ is a projective maximum in the class of almost realcompactifications of X , let (Y, j) be an almost realcompactification of X . By Lemma 2.2 there is a continuous extension of j to a map from ϱX to Y .

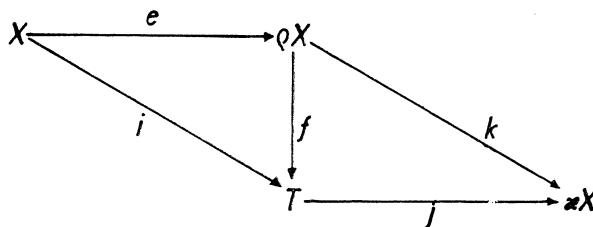
2.4. Theorem. *The following hold:*

- (1) $\varrho X = X$ if and only if X is almost realcompact.
- (2) ϱX is the largest ϱ -extension of X .
- (3) ϱX is the smallest almost realcompact space between X and $\varkappa X$.

Proof (cf. [6, Theorem 3.10]) (1) is immediate from Theorem 2.3 and the construction of ϱX . To see (2) suppose that T is a ϱ -extension of X . By Lemma 2.2 ϱT is also a ϱ -extension of X , so that by the essential uniqueness of ϱX , we have $T \subset \varrho T \approx \varrho X$. For (3), suppose that $X \xrightarrow{i} T \xrightarrow{j} \varkappa T$, where T is almost realcompact. By the projective maximality of $(\varrho X, e)$, there is a continuous function $f: \varrho X \rightarrow T$ such that



commutes. Let $k : \varrho X \rightarrow \varkappa X$ be the inclusion. Since e is a dense map,



commutes. Thus since k and j are inclusions, f must be an inclusion.

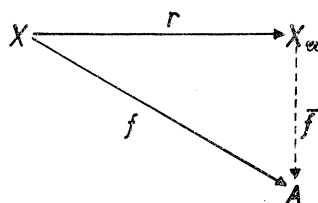
Corollary. *If X is dense in T , then the following are equivalent:*

- (1) T is a ϱ -extension of X .
- (2) T is homeomorphic to a subset of ϱX .
- (3) $\varrho X \approx \varrho T$.

3. EPI-REFLECTIONS

For categorical notations not specifically defined below the reader should see MITCHELL [9]. In all categories under consideration, we will not distinguish among isomorphic objects.

3.1. Definition. A full subcategory \mathfrak{A} of a category \mathfrak{C} is said to be *epi-reflective* in \mathfrak{C} provided that for each object X in \mathfrak{C} , there is an object $X_{\mathfrak{A}}$ in \mathfrak{A} and a \mathfrak{C} -epimorphism $r : X \rightarrow X_{\mathfrak{A}}$ such that for each object A in \mathfrak{A} and each \mathfrak{C} -morphism $f : X \rightarrow A$, there exists a morphism $\bar{f} : X_{\mathfrak{A}} \rightarrow A$ such that



commutes. (Note that since r is an epimorphism, \bar{f} must be unique.)

There are many examples of epi-reflections in general topology. The Stone-Ćech compactification, the Hewitt realcompactification and the Banaschewski zero-dimensional compactification are examples of epi-reflections where all categories in question are full subcategories of the category **Haus** of Hausdorff spaces and con-

tinuous functions. For more examples see [2]. In [3] it was shown that although the category of H -closed spaces and all continuous functions between them is not epi-reflective in the category **Haus**, when the class of morphisms is suitably restricted, an epi-reflective situation does occur, thereby giving rise to a situation analogous to the Stone-Čech compactification. In this section it will be shown that with a similar restriction, the almost realcompactification of the previous section yields an epi-reflective situation analogous to the Hewitt realcompactification.

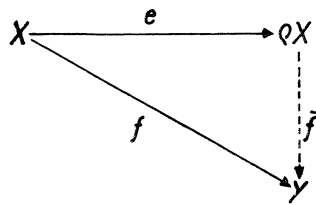
KENNISON [4] has shown that every epi-reflective subcategory of **Haus** must be closed under products and closed subspaces. Thus since closed subspaces of almost realcompact spaces need not be almost realcompact, [1, p. 132] the category of almost realcompact spaces and all continuous functions between them is not epi-reflective in **Haus**.

3.2. Definition. [3] A function $f : X \rightarrow Y$ is said to be *demi-open* (resp. *semi-open*) provided that for each $A \subset X$, $\text{int } A \neq \emptyset \Rightarrow \text{int } (\text{cl } f[A]) \neq \emptyset$ (resp. $\text{int } (A) \neq \emptyset \Rightarrow \text{int } f[A] \neq \emptyset$). Clearly every open map is semi-open, every semi-open map is demi-open, and semi-open and demi-open maps are closed under composition.

3.3. Lemma. *If $X \subset Y$ and $f : X \rightarrow Z$ is demi-open (resp. semi-open), then an extension of f to $\text{cl } X$ must be demi-open (resp. semi-open).*

3.4. Theorem. *The full subcategory of almost realcompact spaces is epi-reflective in the category of Hausdorff spaces and continuous demi-open functions.*

Proof. Let X be any space. Clearly the embedding $e : X \rightarrow \varrho X$ is open, so it is demi-open. Now let Y be almost realcompact and $f : X \rightarrow Y$ be demi-open and continuous. We wish to find a demi-open continuous function $\bar{f} : \varrho X \rightarrow Y$ such that



commutes.

For each $x \in X$, let $\bar{f}(e(x)) = f(x)$. Now if $\mathcal{U} \in \varrho X - e[X]$, then \mathcal{U} is an open ultrafilter on X with c.c.i.p. Since f is demi-open, one can easily show that

$$\hat{\mathcal{U}} = \{W \subset Y : W = \text{int } W, \text{ and for some } U \in \mathcal{U} \text{ int } (\text{cl } W) \supset \text{int } (\text{cl } f[U])\}$$

is an open filter on Y . If W is an open set which meets every member of $\hat{\mathcal{U}}$, then $f^{-1}[W] \cap U \neq \emptyset$ for each $U \in \mathcal{U}$. Thus since \mathcal{U} is maximal, $f^{-1}[W] \in \mathcal{U}$. Conse-

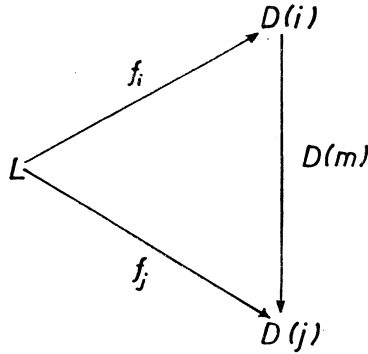
quently, since $\text{int}(\text{cl } W) \supset \text{int}(\text{cl}(f(f^{-1}[W])))$, W must be in $\hat{\mathcal{U}}$. Therefore $\hat{\mathcal{U}}$ is an open ultrafilter.

We wish to show that $\hat{\mathcal{U}}$ has c.c.i.p. If $V_n \in \hat{\mathcal{U}}$, $n = 1, 2, \dots$, then for each n , there is some U_n such that $\text{int}(\text{cl } V_n) \supset \text{int}(\text{cl } f[U_n])$. Since \mathcal{U} has c.c.i.p., there is some $x \in \bigcap_n \text{cl } U_n$. If $f(x) \in W = \text{int } W$, then $f^{-1}[W]$ meets each U_n . Since f is demi-open, W meets each $\text{int}(\text{cl } f[U_n])$. Thus $f(x) \in \bigcap_n \text{cl}(\text{int}(\text{cl } f[U_n])) \subset \bigcap_n \text{cl}(V_n)$.

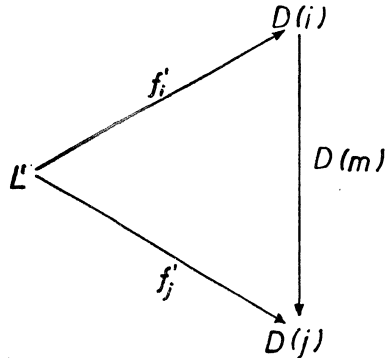
Since $\hat{\mathcal{U}}$ has c.c.i.p. and Y is almost realcompact, $\hat{\mathcal{U}}$ must converge to some point $u \in Y$. For each such $\mathcal{U} \in \mathcal{Q}X - e[X]$, we let $\bar{f}(\mathcal{U}) = u$. The proof of continuity of \bar{f} is essentially the same as that given in Lemma 2.2. \bar{f} is demi-open by Lemma 3.3.

3.5. Corollary. *The full subcategory of almost realcompact spaces is epi-reflective in the category of Hausdorff spaces and continuous semi-open functions.*

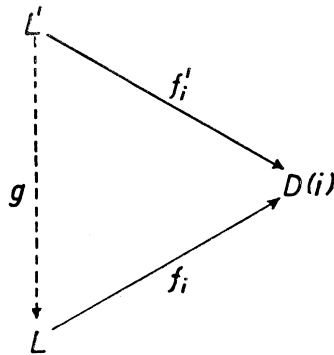
3.6. Definition. A diagram in a category \mathfrak{A} is a functor $D : I \rightarrow \mathfrak{A}$, where I is a small category. A *limit* of a diagram $D : I \rightarrow \mathfrak{A}$ is an object L together with morphisms $L \xrightarrow{f_i} D(i)$ such that for each morphism $m : i \rightarrow j$ in I ,



commutes, and whenever $\{L \xrightarrow{f'_i} D(i)\}$ has the property that for each $m : i \rightarrow j$ in I ,



commutes, then there is a unique morphism $g : L' \rightarrow L$ such that for each i in I

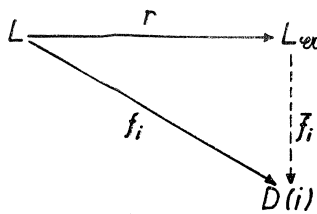


commutes.

It is well-known that epi-reflections are coadjoints of limit preserving inclusions [9, p. 129]. However, this fact cannot be applied in our situation since, for example, the category of Hausdorff spaces and continuous demi-open (or semi-open) functions does not have products. The following strengthened version of the limit preservation property will be useful.

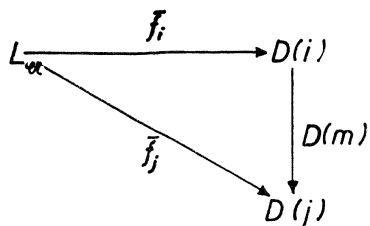
3.7. Theorem. *Let \mathfrak{A} be an epi-reflective subcategory of \mathfrak{B} where \mathfrak{B} is subcategory of \mathfrak{C} having the same objects as \mathfrak{C} and the property that every epimorphism in \mathfrak{B} is an epimorphism in \mathfrak{C} . If $\{L \xrightarrow{f_i} D(i)\}$ is a limit in \mathfrak{C} of a diagram $D : I \rightarrow \mathfrak{A}$, and if each f_i is in \mathfrak{B} , then L must be in \mathfrak{A} .*

Proof. Since \mathfrak{A} is epi-reflective in \mathfrak{B} there is an \mathfrak{A} -object $L_{\mathfrak{A}}$ and a \mathfrak{B} -epimorphism $r : L \rightarrow L_{\mathfrak{A}}$. Since each $D(i)$ is in \mathfrak{A} and $f_i : L \rightarrow D(i)$ is in \mathfrak{B} , there exists a morphism \tilde{f}_i such that the diagram

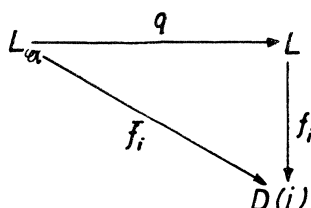


commutes.

Clearly, since r is an epimorphism, for each $m : i \rightarrow j$ in I



commutes. Thus there is a \mathfrak{C} -morphism $q : L_{\mathfrak{A}} \rightarrow L$ such that



commutes for each i . Thus for each i , $f_i \cdot q \cdot r = \bar{f}_i \cdot r = f_i = f_i \cdot 1_L$. Hence by the uniqueness in the definition of limit, $q \cdot r = 1_L$. Also $r \cdot q \cdot r = r \cdot 1_L = 1_{L_{\mathfrak{A}}} \cdot r$, so that since r is an epimorphism in \mathfrak{C} , $r \cdot q = 1_{L_{\mathfrak{A}}}$. Thus r is an isomorphism and consequently L is in \mathfrak{A} .

3.8. Corollary. [Frolík]. *Every product of almost realcompact spaces is almost realcompact and every regular-closed subspace of an almost realcompact space is almost realcompact.*

Proof. Let \mathfrak{B} be the category of Hausdorff spaces and continuous demi-open functions. Let \mathfrak{A} be the full subcategory of \mathfrak{B} consisting of all almost realcompact spaces, and let \mathfrak{C} be **Haus**.

Every topological product $\{P \xrightarrow{\pi_i} D(i)\}$ of almost realcompact spaces is a limit of a diagram D from a small discrete category into \mathfrak{A} and each projection π_i is open and thus is demi-open. Hence by the theorem, P must be in \mathfrak{A} .

A regular-closed subspace S of an almost realcompact space is an equalizer of demi-open maps, i.e., the limit of a diagram $D : I \rightarrow \mathfrak{A}$ where $I = i \xrightarrow[n]{m} j$, and $D(m)$ and $D(n)$ are demi open.

Also a regular closed embedding is demi-open. Thus by the theorem, S must be in \mathfrak{A} .

3.9. Corollary. *If $\{X_{\alpha}, \mu_{\beta\alpha}\}$ is an inverse limit spectrum over a directed set, where each X_{α} is almost realcompact and all spectrum and projection maps are demi-open and continuous, then the inverse limit $\varprojlim X_{\alpha}$ must be almost realcompact.*

Proof. Let \mathfrak{A} , \mathfrak{B} and \mathfrak{C} be the same categories as in the above corollary. Since inverse limits are particular limits, the result is an immediate consequence of the theorem.

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