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# COMPLETE PRIME IDEALS OF BOOLEAN RINGS

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In this paper necessary and sufficient conditions are given for a proper ideal $P$ of a Boolean ring $B$ to be suprema preserving, as well as, for $P$ to be complete. In particular, it is shown that if $P$ is complete then so is $B$.

We recall that a Boolean ring is a ring $B$ such that $x^2 = x$ for every $x \in B$ (thus, $B$ is commutative and has characteristic 2), and that $\leqq$ is a partial order in $B$ where $\leqq$ is defined by:

(1) $$x \leqq y \quad \text{if and only if} \quad xy = x$$

for every element $x$ and $y$ of $B$.

In what follows, any reference to order in a Boolean ring $B$ is made in connection with $\leqq$ as given by (1).

A nonzero element a of a Boolean ring $B$ is called an *atom* [1, p. 27] of $B$ if and only if for every element $x$ of $B$,

(2) $$xa \neq 0 \quad \text{implies} \quad xa = a$$

i.e., if and only if for every element $x$ of $B$,

(3) $$x < a \quad \text{implies} \quad x = 0\,.$$

A Boolean ring need not have a multiplicative unit. If it does then it is called a *Boolean algebra*.

If $H$ is a subset of a Boolean ring $B$ and $m$ is an element of $B$, we let $mH$ denote the subset $\{mx \mid x \in H\}$ of $B$. Clearly, $mB$ is an ideal (and a fortiori, a subring) of $B$.

In what follows we make use of the fact that a proper prime ideal $P$ of a Boolean ring $B$ is a maximal ideal of $B$. This is because the quotient $B/P$ is a Boolean ring with more than one element and without a divosor of zero and the only such Boolean ring is the two-element field.

**Theorem 1.** *Let $P$ be a proper prime ideal of a Boolean ring $B$. Then* $\sup P$ *exists if and only if $B$ has a unit, i.e., if and only if $B$ is a Boolean algebra.*

Proof. Let $\sup P$ exist. If $x \leqq \sup P$, for every $x \in B$ then clearly, $\sup P$ is the unit of $B$. If there exists an element $m$ of $B$ such that $m \nleqq \sup P$ then we consider $u = m + \sup P + m \sup P$. Since $m \neq m \sup P$, we see that $u > \sup P$. But then since $P$ is a proper prime ideal of $B$, the ideal generated by $u$ is equal to $B$, i.e., $\{x \mid x \leqq u\} = B$. Obviously, this implies that $u$ is the unit of $B$. Thus, indeed, $B$ has a unit.

Conversely, let 1 be the unit of $B$. Clearly, 1 is an upper bound of $P$. If $1 \neq \sup P$ then there exists an upper bound $u$ of $P$ such that $u \neq 1$. But then, the subset $uB$ of $B$ is a proper ideal of $B$ containing the proper prime ideal $P$ and therefore $uB = P$. However, since $u \in B$, we see that $u \in P$ and therefore $u = \sup P$. Thus, $\sup P$ exists.

**Theorem 2.** *Let $P$ be a proper prime ideal of a Boolean algebra $A$ with unit 1. Then the following statements are pairwise equivalent.*

$$1 \neq \sup P \,. \tag{4}$$

$$(\sup P) \in P \,. \tag{5}$$

$$1 + \sup P \quad \text{is an atom of} \quad A \,. \tag{6}$$

Proof. In view of Theorem 1, we see that $\sup P$ exists. Let $1 \neq \sup P$ and $u = \sup P$. Then $u \in P$, as shown in the second half of the Proof of Theorem 1. This shows that (4) implies (5). Next, let $(\sup P) \in P$. But then $(1 + \sup P) \in (A - P)$. Moreover, by DeMorgan's law $(1 + \sup P) = \inf (A - P)$ and therefore $(1 + \sup P) = \min (A - P)$. However, $A - P$ is a filter (in fact, an ultrafilter) of $A$ and therefore, for every $x \in A$ if $x < \min (A - P)$ then $x = 0$. But this, in view of (3), shows that $\min (A - P) = 1 + \sup P$ is an atom of $A$. Thus, (5) implies (6). Finally, if $1 + \sup P$ is an atom of $A$ then $(1 + \sup P) \neq 0$ and hence $1 \neq \sup P$. Consequently, (6) implies (4) and the Theorem is proved.

**Corollary 1.** *Let $P$ be a proper prime ideal of a Boolean algebra $A$ with unit 1. Then* $\sup P = 1$ *if and only if $P$ contains all the atoms of $A$.*

Proof. Clearly, it is enough to show that $1 \neq \sup P$ if and only if there exists an atom $a$ such that $a \in (A - P)$. But this follows readily from (4), (5) and (6) since $(\sup P) \in P$ implies $(1 + \sup P) \in (A - P)$.

**Lemma 1.** *Let $m$ and $a$ be elements of a Boolean ring $B$. Then $ma$ is an atom of the subring $mB$ if and only if $ma$ is an atom of $B$.*

Proof. Let $ma$ be an atom of the subring $mB$, and, let $xma \neq 0$ for some element $x$ of $B$. Hence, $mxma \neq 0$ and since $ma$ is an atom of $mB$, from (2) it follows that

$mxma = xma = ma$. Thus, again, in view of (2), we see that $ma$ is an atom of $B$. The converse is obvious.

As expected, a subset $H$ of a Boolean ring is called *suprema preserving* if and only if for every subset $S$ of $H$, if sup $S$ exists then $(\sup S) \in H$.

**Theorem 3.** *Let $P$ be a proper prime ideal of a Boolean ring $B$. Then $P$ is suprema preserving if and only if $B$ has an atom $a$ and $a \notin P$.*

Proof. Let $P$ be suprema preserving and $m \in (B - P)$. But then $m \neq \sup P$. Hence, the unit $m$ of the Boolean algebra $mB$ is not the supremum of the proper prime ideal $mP$ of $mB$. Consequently, by (4) and (6), we see that $m + \sup mP$ which is equal to $m(m + \sup mP)$ is an atom of $mB$. But then from Lemma 1 it follows that $m + \sup mP$ is an atom of $B$. Moreover, $(m + \sup mP) \notin P$ since otherwise $m + + \sup mP$ would be an element of $P$ contradicting (5).

Conversely, let $a$ be an atom of $B$ and $a \notin P$. Let $S$ be a subset of $P$ and $s = \sup S$. We show that $s \in P$. Assume on the contrary that $s \notin P$. Clearly,

$$aP = \{0\} \quad \text{and} \quad a(B - P) = \{a\}\,. \tag{7}$$

Hence, $(s + a)\,x = x$ for every $x \in S$ which implies that $s + a$ is an upper bound of $S$. But this contradicts the hypothesis that $s = \sup S$, since from (7) it follows that $(s + a) < s$.

Thus, Theorem 3 is proved.

Let us recall that a subset $H$ of a Boolean ring is called *complete* if and only if sup $S$ of every subset $S$ of $H$ exists and $(\sup S) \in H$. Clearly, if $H$ is complete then it is also suprema preserving.

**Theorem 4.** *Let $P$ be a proper prime ideal of a Boolean ring $B$. If $P$ is complete then $B$ is complete.*

Proof. Let $P$ be complete. But then from Theorem 1 it follows that $B$ has a unit 1. Now, let $S$ be a subset of $B$. We show that sup $S$ exists. Clearly,

$$S = (S \cap P) \cup (S - P)\,.$$

Since $P$ is complete $\inf\{1 + x \mid x \in (S - P)\}$ exists and by DeMorgan's law

$$1 + \inf\{1 + x \mid x \in (S - P)\} = 1 + \sup(S - P)\,.$$

Hence, $\sup(S - P)$ exists. But then clearly,

$$\sup(S \cap P) + \sup(S - P) + (\sup(S \cap P))(\sup(S - P))$$

is equal to sup $S$. Hence, sup $S$ exists, as desired.

**Theorem 5.** *Let $P$ be a proper prime ideal of a Boolean ring $B$. Then $P$ is complete if and only if $B$ is complete and has an atom $a$ such that $a \in (B - P)$.*

Proof. Let $P$ be complete. Then by Theorem 4 we see that $B$ is complete and by Theorem 3 we see that $B$ has an atom $a$ such that $a \in (B - P)$. Conversely, let $B$ be complete and have an atom $a$ such that $a \in (B - P)$. But then from Theorem 3 it follows that $P$ is suprema preserving. However, since $B$ is complete we see that $P$ is also complete.

**Corollary 2.** *Let $P$ be a proper prime ideal of a Boolean ring $B$. Then the following statements are pairwise equivalent.*

(8) *$P$ is complete.*

(9) *$P$ is suprema preserving and $B$ is complete.*

(10) *$B$ is complete and has an atom $a$ such that $a \in (B - P)$.*

Proof. (8) implies (9) by virtue of Theorem 4. Also, (9) implies (10) by virtue of Theorem 3. Finally, (10) implies (8) by virtue of Theorem 5.

*Reference*

[1] *Sikorski, R.*, Boolean Rings, Springer-Verlag, 1969.

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