Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 22 (1972), No. 2, 224–229

Persistent URL: `http://dml.cz/dmlcz/101093`

Czechoslovak Mathematical Journal, 22 (97) 1972, Praha

A NOTE ON GREEN'S RELATIONS IN $\mathscr{BQ}$-SEMIGROUPS

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(Received November 6, 1970)

I. INTRODUCTION

The purpose of this paper is to describe the structure of Green's relations on $\mathscr{BQ}$-semigroups, i.e., semigroups in which the bi-ideals and the quasi-ideals coincide. We will divide this discussion into two parts. In the first part we will show (2.13) that an $\mathscr{H}$-class contains an irregular element only when it consists of exactly that element.

In the second part we will show (3.5) that in a $\mathscr{BQ}$-semigroup S, an element $s \in S$ is regular if and only if it is quasiregular. We will also show (3.8) that if S is a $\mathscr{BQ}$-semigroup and $a, b \in S$ with $a \mathscr{D} b$ and $R_a < R_b$ and $L_a < L_b$, then a and b are regular. Finally we will show (3.13) that in a $\mathscr{BQ}$-semigroup any irregular $\mathscr{D}$-class is either an $\mathscr{L}$-class or an $\mathscr{R}$-class.

The notation of CLIFFORD and PRESTON [2] will be used.

II. $\mathscr{H}$-CLASS STRUCTURE OF $\mathscr{BQ}$-SEMIGROUPS

(2.1) Definition. A (non-empty) subset B of a semigroup S is a *bi-ideal* if $B \cup BSB \subseteq B$.

(2.2) Definition. Let S be a semigroup and $x \in S$. Then the *principal bi-ideal*, $B(x)$, *generated by* x is the smallest bi-ideal of S containing x. Clearly $B(x) = x \cup xS^1x$.

(2.3) Definition. A (non-empty) subset Q of a semigroup S is called a *quasi-ideal* if $QS \cap SQ \subseteq Q$.

(2.4) Definition. Let S be a semigroup and $x \in S$. Then the *principal quasi-ideal generated*, $Q(x)$, *by* x is the smallest quasi-ideal of S containing x. Clearly $Q(x) = xS^1 \cap S^1x$.

(2.5) Definition. The *class* $\mathscr{B}\mathscr{Q}$ of semigroups will consist precisely of those semigroups whose sets of bi-ideals and quasi-ideals coincide.

One can easily check the following Lemma.

(2.6) Lemma. [3] *Let S be a semigroup. Then for $x, y \in S$, $x \mathscr{H} y$ if and only if $Q(x) = Q(y)$.*

(2.7) Definition. For $a, b \in S$, a given semigroup, we write $a \mathscr{B} b$ if

1) $a = b$ or

2) there exists $u, v \in S$ such that $aua = b$ and $bvb = a$.

Let B_a denote the $\mathscr{B}$-class containing a.

(2.8) Proposition. [(1.3) Proposition KAPP [4].] The relation $\mathscr{B}$ defined in (2.5) is an equivalence relation, indeed, $\mathscr{B} \subseteq \mathscr{H}$.

(2.9) Lemma. [(1.8) Proposition MIELKE [5].] *Let S be a semigroup. Then for $x, y \in S$, $x \mathscr{B} y$ if and only if $B(x) = B(y)$.*

(2.10) Lemma. *If $S \in \mathscr{B}\mathscr{Q}$, then $\mathscr{B} = \mathscr{H}$ in S.*

Proof. We know (2.8) $\mathscr{B} \subseteq \mathscr{H}$. Let $x \mathscr{H} y$. One easily checks that since S is a $\mathscr{B}\mathscr{Q}$-semigroup, $B(x) = Q(x)$ for all $x \in S$. Applying (2.6), we have $B(x) = Q(x) = Q(y) = B(y)$. Thus by (2.9), $x \mathscr{B} y$ and the result follows.

Although $S \in \mathscr{B}\mathscr{Q}$ implies $\mathscr{B} = \mathscr{H}$, we may have $\mathscr{B} = \mathscr{H}$ and $S \notin \mathscr{B}\mathscr{Q}$.

(2.11) Example. [[4] Example (1.10).] Let $S = \{a, a^2, a^3, 0\}$ where $a^4 = 0$. In this semigroup, $\mathscr{B} = \mathscr{H} = \mathscr{J}$, but $B = \{0, a^2\}$ is a bi-ideal which is not a quasi-ideal, since $\{0, a^2\} S \cap S\{0, a^2\} = S\{0, a^2\} = \{0, a^3\} \nsubseteq B$.

(2.12) Lemma. [(1.11) Corollary Mielke [5].] *Let S be a semigroup and $a \in S$. Then either* i) *a is irregular and $B_a = \{a\}$, or* ii) *a is regular and $B_a = H_a$.*

Combining (2.10) and (2.12) we have:

(2.13) Theorem. *Let $S \in \mathscr{B}\mathscr{Q}$. If H_a is an $\mathscr{H}$-class of S and a* is irregular, then $H_a = \{a\}$.

III. $\mathscr{D}$-CLASS STRUCTURE OF $\mathscr{BQ}$-SEMIGROUPS

In our study of the $\mathscr{D}$, $\mathscr{L}$ and $\mathscr{R}$-relations, we will use the following theorem presented by CALAIS to the Semigroup Symposium at Bratislava, Czechoslovakia (1968).

(3.1) Theorem. [Calais; Reims, France.] *Let S be a semigroup. Let $B(x, y)$ denote the minimal bi-ideal of S containing $x, y \in S$, and let $Q(x, y)$ be the minimal quasi-ideal of S containing x and y. Then $S \in \mathscr{BQ}$ if and only if $B(x, y) = Q(x, y)$.*

It is easily seen that $B(x, y) = \{x, y\} \cup xS^1x \cup yS^1y \cup xS^1y \cup yS^1x$, and that $Q(x, y) = (xS^1 \cap S^1x) \cup (yS^1 \cap S^1y) \cup (xS^1 \cap S^1y) \cup (yS^1 \cap S^1x)$.

In the same paper, Calais speculated that another necessary and sufficient condition for $S \in \mathscr{BQ}$ might be that $BS \cap SB = B^2 \cup BSB$ held for every bi-ideal B of S. The condition is clearly sufficient, but the following example shows that it is not necessary.

(3.2) Example. Let $S = (Z/(4), .)$, the integers modulo 4 under multiplication, $S \in \mathscr{BQ}$. Its only proper ideal of any type is $B = \{\bar{0}, \bar{2}\}$, and $BS \cap SB = \{\bar{0}, \bar{2}\}$, $S \cap S\{\bar{0}, \bar{2}\} = S\{\bar{0}, \bar{2}\} = \{\bar{0}, \bar{2}\}$, but $B^2 \cap BSB = \{\bar{0}, \bar{2}\}^2 \cup \{\bar{0}, \bar{2}\}^2\, S = \{\bar{0}\}$.

(3.3) Definition. A non-zero element, a, of a semigroup S is said to be *quasi-regular* if there exist elements $b, c, d, e \in S$ for which we have $a = baca = adae$. A semigroup is said to be *quasi-regular* if each of its elements is quasi-regular (*c.f.* [1]).

The following proposition generalizes [[2] 2.11 (i)] since regular elements are quasi-regular.

(3.4) Proposition. *Let S be a semigroup. Then if $a \in S$ is a quasi-regular element of S, every element of D_a is quasiregular.*

Proof. Let $a \in S$ be a quasi-regular. We will show that every element of L_a is quasi-regular. Dually, every element of R_a will be quasi-regular, and the result will then follow for D_a.

Suppose a is quasi-regular, then $a = auav = sara$ for some $u, v, r, s \in S$. Let $x \in L_a$, if $x \neq a$, then there are $t_1, t_2 \in S$ such that $a = t_1x$ and $x = t_2a$. We then have $x = t_2a = t_2sara = (t_2st_1)\, x(rt_1)\, x$, and $x = t_2a = (t_2a)\, uav = xu(t_1x)\, v = x(ut_1)\, xv$, hence x is quasi-regular. The result now follows.

(3.5) Lemma. *If $S \in \mathscr{BQ}$ an element $a \in S$ is regular if and only if it is quasi-regular.*

Proof. If a is regular, then there exists $a' \in S$ such that $a = aa'a$. Then $a = aa'a(a'a) = (aa')\, aa'a$ so that a is quasi-regular.

If a is quasi-regular, $a \in SaSa$ and $a \in aSaS$. But aSa is a bi-ideal and since $S \in \mathscr{B}\mathscr{Q}$, aSa is a quasi-ideal. Therefore, $a \in (aSa)\, S \cap S(aSa) \subseteq aSa$. Whence a is regular.

(3.6) Proposition. *Let $S \in \mathscr{B}\mathscr{Q}$, then S is regular if and only if S is quasi-regular.*

(3.7) Definition. We partially order the $\mathscr{L}$-classes and $\mathscr{R}$-classes in the usual fashion: $L_x \leqq L_y$ if $S^1 x \subseteq S^1 y$ and $R_x \leqq R_y$ if $xS^1 \subseteq yS^1$.

(3.8) Theorem. *Let $S \in \mathscr{B}\mathscr{Q}$. If $a, b \in S$ with $a \mathscr{D} b$ and both $L_a < L_b$ and $R_a < R_b$, then b is regular* (i.e., *both a and b are regular*).

Proof. Since $a \mathscr{D} b$ and $R_a \neq R_b$ and $L_a \neq L_b$, there exists $t, s \in S$ such that $t \in R_a \cap L_b$ and $s \in R_b \cap L_a$, where $t \neq a, b$, $s \neq a, b$. Since $R_a < R_b$, $t \in R_a \subseteq aS^1 \subset bS^1$ and $t \in L_b \subseteq S^1 b$, it follows that $t \in bS^1 \cap S^1 b = b \cup bS^1 b$. Every quasi-ideal is a bi-ideal, thus $b \cup bS^1 b \subseteq bS^1 \cap S^1 b$, since $b \cup bS^1 b$ is the smallest bi-ideal containing b. $S \in \mathscr{B}\mathscr{Q}$, thus $b \cup bS^1 b$ is a quasi-ideal containing b, but $bS^1 \cap S^1 b$ is the smallest quasi-ideal containing b, so that $b \cup bS^1 b \supseteq bS^1 \cap S^1 b$. Since $t \neq b$, $t \in bS^1 b$. Similarly, $s \in bS^1 b$. Hence there exists $r_1, r_2 \in S^1$, such that $t = br_1 b$ and $s = br_2 b$. Since $t \in L_b \setminus \{b\}$ and $s \in R_b \setminus \{b\}$, we have $m_1, m_2 \in S$ such that $b = m_1 t = sm_2$. If both $r_1, r_2 \in S$, we have $b = m_1 t = m_1 br_1 b$ and $b = sm_2 = br_2 bm_2$, hence b is quasi-regular, therefore regular. If $r_1 = 1$, then $t = br_1 b = b^2$, $b = m_1 t = m_1 b^2 = m_1 bm_1 b^2 = m_1 b(m_1 b)\, b$ and $b = br_2 bm_2$, therefore b is quasi-regular, hence regular. Similarly, if $r_2 = 1$ and $r_1 \in S$, b is regular. Since $t \neq s$, we cannot have $r_1 = r_2 = 1$ otherwise $t = b^2 = s$, and in every case, we have b is regular.

Using (3.8), we now discuss the restricted partial ordering of $\mathscr{L}$- and $\mathscr{R}$-classes in irregular $\mathscr{D}$-classes.

(3.9) Proposition. *If $S \in \mathscr{B}\mathscr{Q}$ and D_a is an irreglar $\mathscr{D}$-class, then either $aS^1 a \cap D_a \subseteq R_a$, or $aS^1 a \cap D_a \subseteq L_a$.*

Proof. Suppose neither $aS^1 a \cap D_a \subseteq R_a$, nor $aS^1 a \cap D_a \subseteq L_a$. Then we have elements b and c such that $b \in (aS^1 a \cap D_a) \setminus R_a$ and $c \in (aS^1 a \cap D_a) \setminus L_a$. Since $b \mathscr{D} c$, there exists $t \in R_b \cap L_c$, and $R_t = R_b < R_a$ for $b \in aS^1 a \subseteq aS^1$. Furthermore, $L_t = L_c < L_a$ since $c \in aS^1 a \subseteq S^1 a$. Thus by (3.8), a is regular contrary to hypothesis. Therefore we must have either $aS^1 a \cap D_a \subseteq R_a$ or $aS^1 a \cap D_a \subseteq L_a$.

(3.10) Proposition. *If $S \in \mathscr{B}\mathscr{Q}$ and D_a is an irregular $\mathscr{D}$-class, then $aS^1 a \cap D_a \subseteq L_a$ if and only if L_a is minimal among the $\mathscr{L}$-classes of S in D_a.*

Proof. If L_a is a minimal $\mathscr{L}$-class of S in D_a, suppose $b \in aS^1 a \cap D_a$ (if $aS^1 a \cap D_a = \square$ we are done), then $L_b \leqq L_a$ and since L_a is a minimal $\mathscr{L}$-class in D_a, we have $L_b = L_a$ and $aS^1 a \cap D_a \subseteq L_a$.

Suppose $aS^1a \cap D_a \subseteq L_a$. Let $b \in D_a$ with $L_b \leqq L_a$, then there exists $r \in L_b \cap R_a$. Hence $r \in L_b \subseteq S^1b \subseteq S^1a$ and $r \in R_a \subseteq aS^1$; thus $r \in aS^1 \cap S^1a = a \cup aS^1a$, for $S \in \mathcal{B}\mathcal{Q}$. If $r = a$ we are done, for then $L_b = L_r = L_a$. Otherwise $r \in aS^1a \cap D_a \subseteq \subseteq L_a$, $L_r = L_a$ and hence $L_a = L_b$. Thus L_a is a minimal $\mathcal{L}$-class of S in D_a.

We note that if $aS^1a \cap D_a = \square$, then R_a and L_a are both minimal among the $\mathcal{R}$- and $\mathcal{L}$-classes of S in D_a.

Combining (3.9) and (3.10) we get:

(3.11) Corollary. *If $S \in \mathcal{B}\mathcal{Q}$ and D_a is an irregular $\mathcal{D}$-class, then either L_a or R_a is minimal in the set of $\mathcal{L}$- or $\mathcal{R}$-classes of S in D_a respectively.*

(3.12) Lemma. *If $S \in \mathcal{B}\mathcal{Q}$ and D is an irregular $\mathcal{D}$-class, then for any two $a, b \in D$, either L_a and L_b are minimal in the set of $\mathcal{L}$-classes of S in D, or R_a and R_b are minimal in the set of $\mathcal{R}$-classes of S in D.*

Proof. For $x \in D$ we know that either L_x is minimal among the $\mathcal{L}$-classes of D, or R_x is minimal among the $\mathcal{R}$-classes of D. Let $a, b \in D$, and suppose to the contrary that L_a and R_b are minimal while neither L_b nor R_a is minimal in the restricted partial ordering. Since L_b is not minimal, there exists $u \in D$ such that $L_u < L_b$, and similarly there exists $v \in D$ such that $R_v < R_a$. Let $t \in L_u \cap R_v$ and $r \in L_b \cap R_a$, then $L_t = = L_u < L_b = L_r$ and $R_t = R_v < R_a = R_r$. Therefore t is regular by (3.8), a contradiction since D is an irregular $\mathcal{D}$-class. Thus either both L_a and L_b are minimal, or both R_a and R_b are minimal.

(3.13) Theorem. *Let $S \in \mathcal{B}\mathcal{Q}$ and D_a be an irregular $\mathcal{D}$-class of S. Then either $D_a = L_a$ or $D_a = R_a$.*

Proof. If $D_a \neq L_a$ and $D_a \neq R_a$, then there is an element $b \in D_a$ such that $L_b \neq L_a$ and $R_b \neq R_a$. By (3.12), either both R_a and R_b are minimal among the $\mathcal{R}$-classes of S in D_a, or both L_a and L_b are minimal among the $\mathcal{L}$-classes of S in D_a. Assume R_a and R_b are minimal. Since $S \in \mathcal{B}\mathcal{Q}$ we have:

$$(*)\qquad \{a, b\} \cup aS^1a \cup bS^1b \cup aS^1b \cup bS^1a = B(a, b) = Q(a, b) =$$
$$= (aS^1 \cap S^1a) \cup (bS^1 \cap S^1b) \cup (aS^1 \cap S^1b) \cup (bS^1 \cap S^1a).$$

Let $u \in R_a \cap L_b$ and $r \in R_b \cap L_a$. Clearly we must have $u, r \notin \{a, b\}$, and $r \neq u$. Then $u \in aS^1 \cap S^1b$ so $u \in B(a, b)$. We examine (*). Since u is not regular, $u \notin uS^1u = = aS^1b$. If $u \in bS^1a$ or bS^1b, then $R_a = R_u \leqq R_b$, and since R_b is minimal, $R_a = R_b$, contrary to our assumption. Thus $u \in aS^1a$ and $L_b = L_u \leqq L_a$. Similarly, $r \in bS^1b$ and $L_a = L_r \leqq L_b$. Thus $L_a = L_b$, contrary to our assumption. Hence if $c \in D_a$, either $c \in L_a$ or $c \in R_a$, and we have $D_a = R_a \cup L_a$.

Suppose $u \in R_a \setminus \{a\}$ and $v \in L_a \setminus \{a\}$, then let $w \in R_u \cap L_v \subseteq D_a = L_a \cup R_a$. Now either $R_v = R_a$ or $L_u = L_a$, and thus either $\{v\} = R_v \cap L_v = R_a \cap L_a = \{a\}$

or $\{u\} = R_u \cap L_u = R_a \cap L_a = \{a\}$, contrary to the hypothesis that $u \in R_a \setminus \{a\}$ and $v \in L_a \setminus \{a\}$. Thus either $R_a \setminus \{a\} = \square$ or $L_a \setminus \{a\} = \square$, and therefore either $D_a = L_a$ or $D_a = R_a$.

Within a $\mathscr{B}\mathscr{Q}$-semigroup, one irregular $\mathscr{D}$-class may be an $\mathscr{L}$-class, and another irregular $\mathscr{D}$-class may be an $\mathscr{R}$-class as in the following example:

(3.14) Example. Let D_1 be a Baer-Levi Semigroup [[2] § 8.1] of all one-to-one mappings, a, of an infinite countable set I into itself such that $I \setminus Ia$ is infinite. D_1 is a right simple irregular semigroup. Let D_1^* be an anti-isomorphic copy of D_1. D_1^* is a left simple irregular semigroup. Let S be the 0-direct union of D_1 and D_1^* where 0 is not in D_1 or D_1^*. S is clearly a semigroup and D_1 and D_1^* are irregular $\mathscr{D}$-classes. Using (3.1), one can check that $S \in \mathscr{B}\mathscr{Q}$, and finally, D_1 is an $\mathscr{R}$-class of S and D_1^* is an $\mathscr{L}$-class of S.

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