Bruce W. Mielke
A note on Green’s relations in $BQ$-semigroups


Persistent URL: [http://dml.cz/dmlcz/101093](http://dml.cz/dmlcz/101093)

Terms of use:

© Institute of Mathematics AS CR, 1972

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.
A NOTE ON GREEN'S RELATIONS IN $\mathcal{B}_2$-SEMIGROUPS

BRUCE W. MIELKE, Providence
(Received November 6, 1970)

I. INTRODUCTION

The purpose of this paper is to describe the structure of Green’s relations on $\mathcal{B}_2$-semigroups, i.e., semigroups in which the bi-ideals and the quasi-ideals coincide. We will divide this discussion into two parts. In the first part we will show (2.13) that an $H$-class contains an irregular element only when it consists of exactly that element. In the second part we will show (3.5) that in a $\mathcal{B}_2$-semigroup $S$, an element $s \in S$ is regular if and only if it is quasiregular. We will also show (3.8) that if $S$ is a $\mathcal{B}_2$-semigroup and $a, b \in S$ with $a \not\preceq b$ and $R_a < R_b$ and $L_a < L_b$, then $a$ and $b$ are regular. Finally we will show (3.13) that in a $\mathcal{B}_2$-semigroup any irregular $D$-class is either an $L$-class or an $R$-class.

The notation of CLIFFORD and PRESTON [2] will be used.

II. $H$-CLASS STRUCTURE OF $\mathcal{B}_2$-SEMIGROUPS

(2.1) Definition. A (non-empty) subset $B$ of a semigroup $S$ is a bi-ideal if $B \cup \cup BSB \subseteq B$.

(2.2) Definition. Let $S$ be a semigroup and $x \in S$. Then the principal bi-ideal, $B(x)$, generated by $x$ is the smallest bi-ideal of $S$ containing $x$. Clearly $B(x) = x \cup xS^1x$.

(2.3) Definition. A (non-empty) subset $Q$ of a semigroup $S$ is called a quasi-ideal if $QS \cap SQ \subseteq Q$.

(2.4) Definition. Let $S$ be a semigroup and $x \in S$. Then the principal quasi-ideal generated, $Q(x)$, by $x$ is the smallest quasi-ideal of $S$ containing $x$. Clearly $Q(x) = xS^1 \cap S^1x$. 
(2.5) Definition. The class $\mathcal{B}2$ of semigroups will consist precisely of those semigroups whose sets of bi-ideals and quasi-ideals coincide.

One can easily check the following Lemma.

(2.6) Lemma. [3] Let $S$ be a semigroup. Then for $x, y \in S$, $x \not\equiv y$ if and only if $Q(x) = Q(y)$.

(2.7) Definition. For $a, b \in S$, a given semigroup, we write $a \mathcal{B} b$ if

1) $a = b$ or

2) there exists $u, v \in S$ such that $aua = b$ and $bvb = a$.

Let $B_a$ denote the $\mathcal{B}$-class containing $a$.

(2.8) Proposition. [(1.3) Proposition KAPP [4].] The relation $\mathcal{B}$ defined in (2.5) is an equivalence relation, indeed, $\mathcal{B} \subseteq \mathcal{H}$.

(2.9) Lemma. [(1.8) Proposition MIELKE [5].] Let $S$ be a semigroup. Then for $x, y \in S$, $x \mathcal{B} y$ if and only if $B(x) = B(y)$.

(2.10) Lemma. If $S \in \mathcal{B}2$, then $\mathcal{B} = \mathcal{H}$ in $S$.

Proof. We know (2.8) $\mathcal{B} \subseteq \mathcal{H}$. Let $x \not\equiv y$. One easily checks that since $S$ is a $\mathcal{B}2$-semigroup, $B(x) = Q(x)$ for all $x \in S$. Applying (2.6), we have $B(x) = Q(x) = Q(y) = B(y)$. Thus by (2.9), $x \mathcal{B} y$ and the result follows.

Although $S \in \mathcal{B}2$ implies $\mathcal{B} = \mathcal{H}$, we may have $\mathcal{B} = \mathcal{H}$ and $S \not\in \mathcal{B}2$.

(2.11) Example. [[4] Example (1.10).] Let $S = \{a, a^2, a^3, 0\}$ where $a^4 = 0$. In this semigroup, $\mathcal{B} = \mathcal{H} = \mathcal{J}$, but $B = \{0, a^2\}$ is a bi-ideal which is not a quasi-ideal, since $\{0, a^2\} \cap S\{0, a^2\} = S\{0, a^2\} = \{0, a^2\} \not\subseteq B$.

(2.12) Lemma. [(1.11) Corollary Mielke [5].] Let $S$ be a semigroup and $a \in S$. Then either i) $a$ is irregular and $B_a = \{a\}$, or ii) $a$ is regular and $B_a = \mathcal{H}_a$.

Combining (2.10) and (2.12) we have:

(2.13) Theorem. Let $S \in \mathcal{B}2$. If $H_a$ is an $\mathcal{H}$-class of $S$ and $a$ is irregular, then $H_a = \{a\}$.
In our study of the \( \mathcal{D} \), \( \mathcal{L} \) and \( \mathcal{B} \)-relations, we will use the following theorem presented by Calais to the Semigroup Symposium at Bratislava, Czechoslovakia (1968).

**Theorem.** [Calais; Reims, France.] Let \( S \) be a semigroup. Let \( B(x, y) \) denote the minimal bi-ideal of \( S \) containing \( x, y \in S \), and let \( Q(x, y) \) be the minimal quasi-ideal of \( S \) containing \( x \) and \( y \). Then \( S \in \mathcal{B}_2 \) if and only if \( B(x, y) = Q(x, y) \).

It is easily seen that \( B(x, y) = \{x, y\} \cup xS^1y \cup yS^1x \), and that \( Q(x, y) = (xS^1 \cap S^1x) \cup (yS^1 \cap S^1y) \cup (yS^1 \cap S^1x) \).

In the same paper, Calais speculated that another necessary and sufficient condition for \( S \in \mathcal{B}_2 \) might be that \( BS \cap SB = B \cap BS \) held for every bi-ideal \( B \) of \( S \). The condition is clearly sufficient, but the following example shows that it is not necessary.

**Example.** Let \( S = (\mathbb{Z}/4, \cdot) \), the integers modulo 4 under multiplication, \( S \in \mathcal{B}_2 \). Its only proper ideal of any type is \( B = \{0, 2\} \), and \( BS \cap SB = \{0, 2\} \), \( S \cap S\{0, 2\} = S\{0, 2\} = \{0, 2\} \), but \( B^2 \cap BSB = \{0, 2\}^2 \cup \{0, 2\}^2 S = \{0\} \).

**Definition.** A non-zero element, \( a \), of a semigroup \( S \) is said to be quasi-regular if there exist elements \( b, c, d, e \in S \) for which we have \( a = baca = adae \). A semigroup is said to be quasi-regular if each of its elements is quasi-regular (c.f. [1]).

The following proposition generalizes [[2] 2.11 (i)] since regular elements are quasi-regular.

**Proposition.** Let \( S \) be a semigroup. Then if \( a \in S \) is a quasi-regular element of \( S \), every element of \( D_a \) is quasi-regular.

**Proof.** Let \( a \in S \) be a quasi-regular. We will show that every element of \( L_a \) is quasi-regular. Dually, every element of \( R_a \) will be quasi-regular, and the result will then follow for \( D_a \).

Suppose \( a \) is quasi-regular, then \( a = auav = sara \) for some \( u, v, r, s \in S \). Let \( x \in L_a \), if \( x \neq a \), then there are \( t_1, t_2 \in S \) such that \( a = t_1x \) and \( x = t_2a \). We then have \( x = t_2a = t_2sara = (t_2st_1)x(rt_1)x \), and \( x = t_2a = (t_2a)uav = xu(t_1)xv = x(ut_1)xv \), hence \( x \) is quasi-regular. The result now follows.

**Lemma.** If \( S \in \mathcal{B}_2 \) an element \( a \in S \) is regular if and only if it is quasi-regular.

**Proof.** If \( a \) is regular, then there exists \( a' \in S \) such that \( a = aa'a \). Then \( a = a'a'a(a'a) = (aa')aa'a \) so that \( a \) is quasi-regular.
If \( a \) is quasi-regular, \( a \in SaSa \) and \( a \in aSaS \). But \( aSa \) is a bi-ideal and since \( S \in \mathcal{B} \), \( aSa \) is a quasi-ideal. Therefore, \( a \in (aSa) S \cap S(aSa) \subseteq aSa \). Whence \( a \) is regular.

**Proposition (3.6).** Let \( S \in \mathcal{B} \), then \( S \) is regular if and only if \( S \) is quasi-regular.

**Definition (3.7).** We partially order the \( L \)-classes and \( R \)-classes in the usual fashion: \( L_x \leq L_y \) if \( S^1 x \leq S^1 y \) and \( R_x \leq R_y \) if \( x S^1 \leq y S^1 \).

**Theorem (3.8).** Let \( S \in \mathcal{B} \). If \( a, b \in S \) with \( a \not\sim b \) and both \( L_a \prec L_b \) and \( R_a \prec R_b \), then \( b \) is regular (i.e., both \( a \) and \( b \) are regular).

**Proof.** Since \( a \not\sim b \) and \( R_a \neq R_b \) and \( L_a \neq L_b \), there exists \( t, s \in S \) such that \( t \in R_a \cap L_b \) and \( s \in R_b \cap L_a \), where \( t + a, b, s + a, b \). Since \( R_a \prec R_b \), \( t \in R_a \subseteq a S^1 \subseteq b S^1 \) and \( t \in L_b \subseteq S^1 b \), it follows that \( t \in b S^1 \cap S^1 b = b \cup b S^1 b \). Every quasi-ideal is a bi-ideal, thus \( b \cup b S^1 \) is the smallest bi-ideal containing \( b \). \( S \in \mathcal{B} \), thus \( b \cup b S^1 b \) is a quasi-ideal containing \( b \), but \( b S^1 \cap S^1 b \) is the smallest quasi-ideal containing \( b \), so that \( b \cup b S^1 b \supseteq b S^1 \cap S^1 b \). Since \( t + b, t \in b S^1 b \). Similarly, \( s \in b S^1 b \). Hence there exists \( r_1, r_2 \in S^1 \), such that \( t = b r_1 b \) and \( s = b r_2 b \). Since \( t \in L_a \setminus \{ b \} \) and \( s \in R_b \setminus \{ b \} \), we have \( m_1, m_2 \in S \) such that \( b = m_1 t = t m_2 \). If both \( r_1, r_2 \in S \), we have \( b = m_1 t = m_1 b r_1 b \) and \( b = s m_2 = b r_2 b m_2 \), hence \( b \) is quasi-regular, therefore regular. If \( r_1 = 1 \), then \( t = b r_1 b = b^2 \), \( b = m_1 t = m_1 b^2 = m_1 b m_1 b^2 = m_1 b (m_1 b) b \) and \( b = b r_2 b m_2 \), therefore \( b \) is quasi-regular, hence regular. Similarly, if \( r_2 = 1 \) and \( r_1 \in S \), \( b \) is regular. Since \( t + s \), we cannot have \( r_1 = r_2 = 1 \) otherwise \( t = b^2 = s \), and in every case, we have \( b \) is regular.

Using (3.8), we now discuss the restricted partial ordering of \( \mathcal{L} \)- and \( \mathcal{R} \)-classes in irregular \( \mathcal{D} \)-classes.

**Proposition (3.9).** If \( S \in \mathcal{B} \) and \( D_a \) is an irregular \( \mathcal{D} \)-class, then either \( a S^1 a \cap \cap D_a \subseteq R_a \) or \( a S^1 a \cap \cap D_a \subseteq L_a \).

**Proof.** Suppose neither \( a S^1 a \cap \cap D_a \subseteq R_a \) nor \( a S^1 a \cap \cap D_a \subseteq L_a \). Then we have elements \( b \) and \( c \) such that \( b \in (a S^1 a \cap \cap D_a) \setminus R_a \) and \( c \in (a S^1 a \cap \cap D_a) \setminus L_a \). Since \( b \not\sim c \), there exists \( t \in R_a \cap L_c \), and \( r_t \prec R_a \prec R_a \) for \( b \in a S^1 a \subseteq a S^1 b \). Furthermore, \( L_t \prec L_a \) since \( c \in a S^1 a \subseteq S^1 a \). Thus by (3.8), \( a \) is regular contrary to hypothesis. Therefore we must have either \( a S^1 a \cap \cap D_a \subseteq R_a \) or \( a S^1 a \cap \cap D_a \subseteq L_a \).

**Proposition (3.10).** If \( S \in \mathcal{B} \) and \( D_a \) is an irregular \( \mathcal{D} \)-class, then \( a S^1 a \cap \cap D_a \subseteq L_a \) if and only if \( L_a \) is minimal among the \( \mathcal{L} \)-classes of \( S \) in \( D_a \).

**Proof.** If \( L_a \) is a minimal \( \mathcal{L} \)-class of \( S \) in \( D_a \), suppose \( b \in a S^1 a \cap \cap D_a \) (if \( a S^1 a \cap \cap D_a = \emptyset \), we are done), then \( L_b \subseteq L_a \) and since \( L_a \) is a minimal \( \mathcal{L} \)-class in \( D_a \), we have \( L_b = L_a \) and \( a S^1 a \cap \cap D_a \subseteq L_a \).
Suppose \(aS^1a \cap D_a \subseteq L_a\). Let \(b \in D_a\) with \(L_b \subseteq L_a\), then there exists \(r \in L_b \cap R_a\). Hence \(r \in L_a \subseteq S^1b \subseteq S^1a\) and \(r \in R_a \subseteq aS^1\); thus \(r \in aS^1 \cap S^1a = a \cup aS^1a\), for \(S \in \mathcal{B}_2\). If \(r = a\) we are done, for then \(L_b = L_a\). Otherwise \(r \in aS^1a \cap D_a \subseteq L_a\), \(L_r = L_a\) and hence \(L_a = L_g\). Thus \(L_a\) is a minimal \(L\)-class of \(S\) in \(D_a\).

We note that if \(aS^1a \cap D_a = \emptyset\), then \(R_a\) and \(L_a\) are both minimal among the \(\mathcal{R}\)- and \(\mathcal{L}\)-classes of \(S\) in \(D_a\).

Combining (3.9) and (3.10) we get:

(3.11) **Corollary.** If \(S \in \mathcal{B}_2\) and \(D_a\) is an irregular \(\mathcal{L}\)-class, then either \(L_a\) or \(R_a\) is minimal in the set of \(\mathcal{L}\)- or \(\mathcal{R}\)-classes of \(S\) in \(D_a\) respectively.

(3.12) **Lemma.** If \(S \in \mathcal{B}_2\) and \(D\) is an irregular \(\mathcal{L}\)-class, then for any two \(a, b \in D\), either \(L_a\) and \(L_b\) are minimal in the set of \(\mathcal{L}\)-classes of \(S\) in \(D\), or \(R_a\) and \(R_b\) are minimal in the set of \(\mathcal{R}\)-classes of \(S\) in \(D\).

**Proof.** For \(x \in D\) we know that either \(L_x\) is minimal among the \(\mathcal{L}\)-classes of \(D\), or \(R_x\) is minimal among the \(\mathcal{R}\)-classes of \(D\). Let \(a, b \in D\), and suppose to the contrary that \(L_a\) and \(R_b\) are minimal while neither \(L_a\) nor \(R_a\) is minimal in the restricted partial ordering. Since \(L_b\) is not minimal, there exists \(u \in D\) such that \(L_u < L_b\), and similarly there exists \(v \in D\) such that \(R_v < R_a\). Let \(t \in L_u \cap R_v\) and \(r \in L_b \cap R_a\), then \(t = L_t = L_u < L_b = L_x\) and \(R_t = R_v < R_a = R_r\). Therefore \(t\) is regular by (3.8), a contradiction since \(D\) is an irregular \(\mathcal{L}\)-class. Thus either both \(L_a\) and \(L_b\) are minimal, or both \(R_a\) and \(R_b\) are minimal.

(3.13) **Theorem.** Let \(S \in \mathcal{B}_2\) and \(D_a\) be an irregular \(\mathcal{L}\)-class of \(S\). Then either \(D_a = L_a\) or \(D_a = R_a\).

**Proof.** If \(D_a \neq L_a\) and \(D_a \neq R_a\), then there is an element \(b \in D_a\) such that \(L_b \neq L_a\) and \(R_b \neq R_a\). By (3.12), either both \(R_a\) and \(R_b\) are minimal among the \(\mathcal{R}\)-classes of \(S\) in \(D_a\), or both \(L_a\) and \(L_b\) are minimal among the \(\mathcal{L}\)-classes of \(S\) in \(D_a\). Assume \(R_a\) and \(R_b\) are minimal. Since \(S \in \mathcal{B}_2\) we have:

\[
\{a, b\} \cup aS^1a \cup bS^1b \cup aS^1b \cup bS^1a = B(a, b) = Q(a, b) = \tag{*} \\
= (aS^1 \cap S^1a) \cup (bS^1 \cap S^1b) \cup (aS^1 \cap S^1b) \cup (bS^1 \cap S^1a) .
\]

Let \(u \in R_a \cap L_b\) and \(r \in R_b \cap L_a\). Clearly we must have \(u, r \notin \{a, b\}\), and \(r \neq u\). Then \(u \in aS^1 \cap S^1b\) so \(u \in B(a, b)\). We examine (*) since \(u\) is not regular, \(u \notin uS^1u = aS^1b\). If \(u \in bS^1a\) or \(bS^1b\), then \(R_a = R_a \leq R_b\), and since \(R_b\) is minimal, \(R_a = R_b\), contrary to our assumption. Thus \(u \in aS^1a\) and \(L_b = L_u \leq L_a\). Similarly, \(r \in bS^1b\) and \(L_a = L_r \leq L_b\). Thus \(L_a = L_b\), contrary to our assumption. Hence if \(c \in D_a\), either \(c \in L_a\) or \(c \in R_a\), and we have \(D_a = R_a \cup L_a\).

Suppose \(u \in R_a \setminus \{a\}\) and \(v \in L_a \setminus \{a\}\), then let \(w \in R_a \cap L_v \subseteq D_a = L_a \cup R_a\).

Now either \(R_w = R_a\) or \(L_w = L_a\), and thus either \(\{v\} = R_v \cap L_w = R_a \cap L_a = \{a\}\),
or \( \{u\} = R_u \cap L_u = R_a \cap L_a = \{a\} \), contrary to the hypothesis that \( u \in R_a \setminus \{a\} \) and \( v \in L_a \setminus \{a\} \). Thus either \( R_a \setminus \{a\} = \varnothing \) or \( L_a \setminus \{a\} = \varnothing \), and therefore either \( D_a = L_a \) or \( D_a = R_a \).

Within a \( \mathcal{B}\)-semigroup, one irregular \( \mathcal{D} \)-class may be an \( \mathcal{L} \)-class, and another irregular \( \mathcal{D} \)-class may be an \( \mathcal{R} \)-class as in the following example:

**Example.** Let \( D_1 \) be a Baer-Levi Semigroup \([2] \S 8.1\) of all one-to-one mappings, \( a \), of an infinite countable set \( I \) into itself such that \( I \setminus Ia \) is infinite. \( D_1 \) is a right simple irregular semigroup. Let \( D^*_1 \) be an anti-isomorphic copy of \( D_1 \). \( D^*_1 \) is a left simple irregular semigroup. Let \( S \) be the 0-direct union of \( D_1 \) and \( D^*_1 \) where 0 is not in \( D_1 \) or \( D^*_1 \). \( S \) is clearly a semigroup and \( D_1 \) and \( D^*_1 \) are irregular \( \mathcal{D} \)-classes. Using (3.1), one can check that \( S \in \mathcal{B}\), and finally, \( D_1 \) is an \( \mathcal{R} \)-class of \( S \) and \( D^*_1 \) is an \( \mathcal{L} \)-class of \( S \).

**Bibliography**


Author's address: Rhode Island College, Providence, Rhode Island 02908 U.S.A.