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ON THE 3-SEGMENT PROPERTY FOR COMPLEX-VALUED  
FUNCTIONS\*)

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1. A function  $f$  from the open upper half plane  $H$  into the Riemann sphere  $W$  is said to have the 3-segment property at the point  $p$  on the real line  $R$  if there exist three segments in  $H$  emanating from  $p$  such that the intersection of the cluster sets of  $f$  at  $p$  relative to these segments is empty. The following question was posed in [1]: does there exist a continuous function from  $H$  into  $W$  having the 3-segment property at each point of a set of positive measure or of second category on  $R$ ? In partial answer to this question, we introduce the 3-segment property relative to two direction functions and show that the set of points at which a continuous function has this property is of first category and measure zero on  $R$ .

2. For  $s \in (0, \pi)$  and  $p \in R$ , the set  $\{z : \arg(z - p) = s\}$  is called a *segment at  $p$*  and  $s$  is said to be its *direction*. A monotone function from  $R$  into the open interval  $(0, \pi)$  which is absolutely continuous on finite intervals is called a *direction function*. Throughout this paper  $\lambda_1$  and  $\lambda_2$  will denote two direction functions, while  $S_1(p)$  and  $S_2(p)$  will denote the segments at  $p \in R$  whose corresponding directions  $\text{dir } S_1(p)$  and  $\text{dir } S_2(p)$  are equal to  $\lambda_1(p)$  and  $\lambda_2(p)$ , respectively.

If  $f$  is a function from  $H$  into  $W$ , the *cluster set*  $CS_j(p)$  of  $f$  at  $p \in R$  relative to the segment  $S_j(p)$  is defined to be the set of all points  $w \in W$  for which there exists a sequence  $\{z_k\} \subset S_j(p)$  with  $z_k \rightarrow p$  and  $f(z_k) \rightarrow w$ .

**Definition.** The function  $f : H \rightarrow W$  has the 3-segment property at  $p \in R$  relative to the direction functions  $\lambda_1$  and  $\lambda_2$  if there exists a segment  $S_3(p)$  at  $p$  distinct from the segments  $S_1(p)$  and  $S_2(p)$  such that

$$CS_1(p) \cap CS_2(p) \cap CS_3(p) = \emptyset.$$

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**Theorem.** If  $f : H \rightarrow W$  is continuous, the set  $Q$  of points at which  $f$  has the 3-segment property relative to the direction functions  $\lambda_1$  and  $\lambda_2$  is of first category and measure zero on  $R$ .

To prove this theorem we make use of the following lemma (for a related result see [2, p. 158]). Before stating the lemma we introduce some terminology: The point  $p \in R$  is a *point of density* of a measurable set  $E \subset R$  if

$$\lim_{a \rightarrow 0^+} \frac{m[E \cap (p - a, p + a)]}{2a} = 1,$$

where  $m$  denotes linear Lebesgue measure; and  $p$  is a *point of right density* of  $E$  if

$$\lim_{a \rightarrow 0^+} \frac{m[E \cap (p, p + a)]}{a} = 1.$$

A measurable function  $g : R \rightarrow R$  is (*right*) *approximately continuous* at  $p^* \in R$  if there exists a set  $E \subset R$  such that (1)  $p^*$  is a point of (right) density of  $E$  and (2)  $f(p) \rightarrow f(p^*)$  as  $p \rightarrow p^*$  provided  $p \in E$ .

**Lemma.** Let  $g : [0, +\infty) \rightarrow [0, +\infty)$  be monotone and absolutely continuous on  $[0, a]$  for some  $a > 0$  with  $g(0) = 0$  and  $g'(0) > 0$ , and let  $g'$  be right approximately continuous at 0. If 0 is a point of right density of a measurable set  $E$ , then 0 is a point of right density of  $g(E)$ .

*Proof.* Choose a set  $F \subset [0, +\infty)$  such that 0 is a point of right density of  $F$  and  $g'(p) \rightarrow g'(0)$  as  $p \rightarrow 0$  provided  $p \in F$ . Set  $E^* = E \cap F$ , and let  $\phi$  denote the characteristic function of  $E^*$ . Then for each  $p \in (0, a)$  we have

$$\begin{aligned} \frac{m[g(E) \cap (0, g(p))]}{g(p)} &\geq \frac{m[g(E^*) \cap (0, g(p))]}{g(p)} = \frac{1}{g(p)} \int_0^{g(p)} \phi(g^{-1}(\sigma)) \, d\sigma = \\ &= \frac{1}{g(p)} \int_0^p \phi(\tau) g'(\tau) \, d\tau = \frac{g'(0)}{g(p)} \int_0^p \phi(\tau) \, d\tau + \frac{1}{g(p)} \int_0^p \phi(\tau) [g'(\tau) - g'(0)] \, d\tau. \end{aligned}$$

Let  $\varepsilon > 0$  be given, and choose a point  $p_\varepsilon < a$  so that (1)  $|g'(\tau) - g'(0)| < \varepsilon g'(0)/2$  for all  $\tau \in [0, p_\varepsilon] \cap E^*$  and (2)  $g(p)/p \geq g'(0)/2$  for all  $p \in (0, p_\varepsilon] \cap E^*$ . Then for all  $p \in (0, p_\varepsilon]$ ,

$$\left| \frac{1}{g(p)} \int_0^p \phi(\tau) [g'(\tau) - g'(0)] \, d\tau \right| < \varepsilon;$$

and so

$$\lim_{p \rightarrow 0^+} \frac{1}{g(p)} \int_0^p \phi(\tau) [g'(\tau) - g'(0)] \, d\tau = 0.$$

Consequently

$$\begin{aligned} \lim_{p \rightarrow 0^+} \frac{m[E \cap (0, g(p))]}{g(p)} &\geq \lim_{p \rightarrow 0^+} \frac{g'(0)}{g(p)} \int_0^p \phi(\tau) \, d\tau = \\ &= g'(0) \lim_{p \rightarrow 0^+} \left( \frac{p}{g(p)} \right) \left( \frac{m[E^* \cap (0, p)]}{p} \right) = 1; \end{aligned}$$

hence the lemma is proved.

**Proof of theorem.** Let  $\mathcal{B}$  be a countable basis for the usual topology on  $W$ , and let  $\mathcal{G}$  be the countable collection of all finite unions of the sets  $B \in \mathcal{B}$ . Then let  $\mathcal{G}^*$  be the set of all 3-tuples  $(G_1, G_2, G_3)$  of sets in  $\mathcal{G}$  for which  $\bar{G}_1 \cap \bar{G}_2 \cap \bar{G}_3 = \emptyset$ , where  $\bar{G}_j$  denotes the closure of  $G_j$ . For each 3-tuple  $(G_1, G_2, G_3)$  in  $\mathcal{G}^*$ , each pair of rational numbers  $\alpha, \beta$  satisfying  $0 < \alpha < \beta < \pi$ , and each positive rational  $r \leq 1$ , let  $Q(G_1, G_2, G_3; \alpha, \beta; r)$  be the set of points  $p \in R$  at which there exists a segment  $S_3(p)$  such that

- (1)  $\alpha \leq \text{dir } S_3(p) \leq \beta$
  - (2)  $\lambda_j(p) \notin (\alpha - r, \beta + r) \quad (j = 1, 2)$
- and
- (3)  $f(S_j(p, r)) \subset \bar{G}_j \quad (j = 1, 2, 3),$

where  $S_j(p, r) = S_j(p) \cap \{z : \text{Im}(z) \leq r\}$ . One can easily see that each of the sets  $Q(G_1, G_2, G_3; \alpha, \beta; r)$  is closed and that  $Q$  is the countable union of all of them. Thus, if we let  $Q_0$  denote the set  $Q(G_1, G_2, G_3; \alpha, \beta; r)$ , it suffices to show that  $Q_0$  is of measure zero on  $R$ .

Let  $Q_0^*$  be the set of points in  $Q_0$  which are points of density of  $Q_0$ , and assume that  $mQ_0^* > 0$ . Since  $\lambda'_j$  ( $j = 1, 2$ ) is approximately continuous a.e. on  $R$ , there exists a point  $p_0 \in Q_0^*$  at which  $\lambda'_1$  and  $\lambda'_2$  are approximately continuous. For convenience we take  $p_0 = 0$  and assume that  $\text{dir } S_3(0) > \text{dir } S_j(0)$  ( $j = 1, 2$ ), where  $S_3(0)$  is the segment at 0 guaranteed by  $0 \in Q_0$ . For  $j = 1, 2$  define the function  $\mu_j : (0, +\infty) \rightarrow S_3(0)$  by

$$\{\mu_j(p)\} = S_3(0) \cap S_j(p).$$

Then define the function  $g_j : [0, +\infty) \rightarrow [0, +\infty)$  by  $g_j(p) = |\mu_j(p)|$  for  $p > 0$  and  $g_j(0) = 0$ . It is easily verified that near 0

$$g_j(p) = p \{ \sin \lambda_j(p) / \sin [\lambda_j(p) - \text{dir } S_3(0)] \}$$

and that  $g_j$  satisfies the hypotheses of the lemma for  $j = 1, 2$ . It follows that 0 is a point of right density for the set  $g_1(Q_0) \cap g_2(Q_0)$ . Consequently there exist points  $p_1, p_2 \in Q_0 \cap (0, +\infty)$  such that

$$\mu_1(p_1) = \mu_2(p_2) \in S_3(0, r).$$

Setting  $z = \mu_1(p_1)$  we have

$$f(z) \in \bar{G}_1 \cap \bar{G}_2 \cap \bar{G}_3$$

which contradicts  $(G_1, G_2, G_3) \in \mathcal{G}^*$ . Thus we must have  $mQ_0^* = 0$ , and it follows from the Lebesgue Density Theorem that  $mQ_0 = 0$ . This completes the proof of the theorem.

Remark. In effect BAGEMIHL, PIRANIAN and YOUNG [1, p. 30] have constructed a continuous function from  $H$  into  $W$  having the 3-segment property relative to the constant direction functions  $\lambda_1(p) \equiv \pi/4$  and  $\lambda_2(p) \equiv \pi/2$  at each point of the Cantor "middle halves" set in  $[0, 1]$ ; hence the set  $Q$  of the above theorem need not be countable.

#### References

- [1] *F. Bagemihl, G. Piranian, and G. S. Young*, Intersections of cluster sets, *Bul. Inst. Politehn. Iași (N.S.)* 5 (9), (1959), no. 3–4, 29–34.
- [2] *J. L. Doob*, The boundary values of analytic functions, *Trans. Amer. Math. Soc.* 34 (1932), 153–170.

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