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SOME REMARKS ON THE LÜROTH EXPANSION

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1. Summary and introduction. Recently much attention was paid to the ergodic properties of the denominators d_n in the Lüroth expansion of real numbers $0 < x \leqq \leqq 1$. It was recognized independently by ŠALÁT [7], JAGER and DEVROEDT [5] and in my work [2] that the d_n are independent random variables on the probability space S where the basic set is the interval $(0, 1)$ and the probability is Lebesgue measure. This fact provides much information concerning the sequence $\{d_n\}$ and different questions were extensively studied in the papers [2], [3], [4], [5] and [7]. This research was induced by the results of Šalát [7], who investigated the order of magnitude of d_n and the aim of the present paper is to give some results in this direction.

The algorithm for the Lüroth expansion of x is as follows. Let $x = x_1$ and let

$$d_n = [1/x_n] + 1\,, \quad x_{n+1} = (x_n - 1/d_n)\, d_n(d_n - 1) \tag{1}$$

where $[y]$ stands for the integer part of y. The algorithm (1) leads to the infinite series expansion

$$x = \frac{1}{d_1} + \frac{1}{s_1}\frac{1}{d_2} + \ldots + \frac{1}{s_1 s_2 \ldots s_{n-1}}\frac{1}{d_n} + \ldots \tag{2}$$

where

$$s_k = d_k(d_k - 1)\,. \tag{3}$$

For the number theoretical aspects of this series expansion see PERRON [6], pp. 116–127.

Let S be the probability space specified above, i.e. the interval $(0, 1)$ being its basic set and the probability $P(.)$ is Lebesgue measure. Evidently, the denominators d_n are functions $d_n(x)$ of x and it is easily seen that the d's are random variables on S (i.e. Lebesgue measurable). We shall quote as Lemma 1 a result of the papers [2], [5] and [7] concerning the distribution of the d's.

Lemma 1. *The denominators $d_1, d_2, \ldots$ are independent random variables on S and the distribution of d_n is given by*

$$P(d_n = t) = \frac{1}{t(t-1)}$$

for $n = 1, 2, \ldots$ and $t \geqq 2$.

In the next section we shall investigate some questions related to the work [7] of Šalát and in the last section, the asymptotic behaviour of

$$D_n = D_n(x) = \max(d_1, d_2, \ldots, d_n) \tag{4}$$

will be studied. The basic tool will be Lemma 1.

2. Extensions of some of the results of Šalát. In this section we shall concentrate on problems investigated in the first two sections of [7]. Some of the results were also obtained in [2] and [5], thus we do not discuss those questions here again.

Let a_n be a sequence of real numbers and put X_n for the indicator variable of the set $\{x : d_n > a_n\}$, i.e.,

$$X_n = \begin{cases} 1 & \text{if } \; d_n > a_n \\ 0 & \text{otherwise}. \end{cases} \tag{5}$$

By Lemma 1, the random variables X_n are independent and

$$P(X_n = 1) = \sum_{j=[a_n]+1}^{+\infty} \frac{1}{j(j-1)} = \frac{1}{[a_n]}, \tag{6}$$

assuming that

$$a_n \geqq 1. \tag{7}$$

This assumption is not a restriction concerning the generality of our investigation, since evidently from (1), $d_n \geqq 2$ for all n. We first prove

Theorem 1. *Let X_n be given by (5) and let (7) be satisfied. Then the series ΣX_n converges almost everywhere if, and only if, $\Sigma\, 1/a_n < +\infty$. On the other hand, if*

$$A_N = \sum_{n=1}^{N} \frac{1}{[a_n]} \to +\infty$$

then, putting

$$Y_N = \sum_{n=1}^{N} X_n \quad \text{and} \quad B_N^2 = \sum_{n=1}^{N} \frac{1}{[a_n]}\left(1 - \frac{1}{[a_n]}\right),$$

we have the following limit relation

$$(8) \qquad P(Y_N - A_N < zB_N) \to (2\pi)^{-1/2} \int_{-\infty}^{z} e^{-(1/2)t^2}\, dt$$

as $N \to +\infty$, *under the additional assumption* $B_N \to +\infty$.

Proof. It immediately follows from (5)–(7) and from the fact that the X's are independent. Namely, the first part is a consequence of the Borel-Cantelli lemma ([1], p. 188) and our conditions guarantee that the central limit theorem under Lindeberg's conditions are applicable to the X's and we obtain the asymptotic normality (8), which terminates the proof (for Lindeberg's theorem, see [1], p. 239).

(8) implies that, in probability, Y_N is asymptotically A_N if both A_N and B_N tend to $+\infty$. Though in the general case much more can not be said, in a special case the following stronger statement applies.

Theorem 2. *Assume that*

$$\lim_{N=+\infty} A_N/N = \alpha$$

exists and is positive. Then for almost all x *in* $(0, 1)$,

$$\lim_{N=+\infty} (X_1 + \ldots + X_N)/N = \alpha$$

The statement remains to hold also if $\alpha = 0$.

Proof. Introduce the random variables $Z_n = X_n - 1/[a_n]$. Putting $E(.)$ and $V(.)$ for the expectation and the variance, respectively, of the random variable in the brackets, we have from (5) and (6) that $E(Z_n) = 0$ and that $V(Z_n) = V(X_n) = (1 - 1/[a_n])/[a_n]$, hence by (7), Kolmogorov's theorem ([1], p. 243) is applicable to the Z's, which yields that for almost all x in $(0, 1)$,

$$\lim_{N=+\infty} (Z_1 + \ldots + Z_N)/N = 0$$

which now gives both parts of our theorem, hence the proof is complete.

Theorem 2 is related to the concept of asymptotic density, which is defined as follows. Let $c_1, c_2, \ldots$ be an increasing sequence of positive integers and let $C(M)$ denote the number of elements of the sequence $\{c_j\}$ for which $c_t \leqq M$. If $C(M)/M$ has limit as $M \to +\infty$, then we say that the sequence $\{c_j\}$ has asymptotic density. Theorem 2 thus gives criterion for the sequence $\{n : d_n > a_n\}$ to have asymptotic density, for almost all x. Note that if $a_n \to +\infty$ then, in Theorem 2, α exists and equals zero, for which case the conclusion of Theorem 2 was obtained in [7]. The well known fact also follows from Theorem 2 that any positive integer $r \geqq 2$ has asymptotic density among the denominators d_n, for almost all x. As a matter of fact, applying Theorem 2 with $a_n = 2$ for all n, then with $a_n = 3$ for all n, and so on, we get successively the densities of $\{n : d_n = 2\}$, $\{n : d_n = 3\}, \ldots$ But this theorem also shows that the converse to the result of Šalát ([7], Theorem 2.4) is not valid, namely, the density of the set $\{n : d_n > a_n\}$ can be zero with liminf a_n being finite and also this density can be positive if limsup $a_n = +\infty$.

By Lemma 1 and by the Borel-Cantelli lemma, we can have a deeper insight into the behaviour of the sequence $\{d_n\}$. The following theorem appears to be of interest in the light of the present investigation.

Theorem 3. *Let $1 < a_n \leqq b_n$ be two sequences of real numbers which tend to $+\infty$. Let us further assume that*

$$\limsup a_n/b_n = u < 1 \tag{9}$$

and that

$$\sum_{n=1}^{+\infty} \frac{1}{a_n b_n} = +\infty\,. \tag{10}$$

Then for almost all x, the inequalities $b_n < d_n \leqq b_n(1 + 1/a_n)$ hold for infinite many values of n.

Proof. Since

$$P(v < d_n \leqq w) = P(v < d_n) - P(w < d_n)$$

we have from (6)

$$\begin{aligned} P(b_n < d_n \leqq b_n + b_n/a_n) &= 1/[b_n] - 1/[b_n + b_n/a_n] \geqq \\ &\geqq 1/b_n - 1/(b_n + b_n/a_n - 1) \geqq (1 - a_n/b_n)/(a_n b_n + b_n) \geqq c/a_n b_n \end{aligned} \tag{11}$$

where c is a suitable constant, which, in view of (9), for n sufficiently large, can be chosen arbitrarily close to $1 - u$, hence $c > 0$. Now, by Lemma 1, the Borel-Cantelli lemma is applicable, hence (10) and (11) imply the conclusion of Theorem 3. The proof is thus complete.

Theorem 3 states that if for a sequence b_n tending to $+\infty$, there is a sequence a_n such that (9) and (10) hold, then, for almost all x, infinitely often $d_n \sim b_n$. This raises the question whether $D_n = \max(d_1, d_2, \ldots, d_n)$ has an asymptotic law. We shall deal with this question in the next section.

3. The distribution of the largest of the first n **denominators.** Let D_n be the maximum of the first n denominators.

Theorem 4. *For any fixed $y > 0$, as $n \to +\infty$,*

$$\lim P(D_n/n \leqq y) = \exp(-1/y)\,. \tag{12}$$

Proof. Define the events $T_j = \{x : d_j/n \leqq y\}$, $1 \leqq j \leqq n$. Evidently

$$\{x : D_n/n \leqq y\} = \bigcap_{j=1}^{n} T_j$$

and therefore by Lemma 1 and by (6)

$$P(D_n/n \leqq y) = \prod_{j=1}^{n} P(d_j/n \leqq y) = \prod_{j=1}^{n} (1 - P(d_j/n > y)) = (1 - 1/[ny])^n$$

which by the well known elementary limit relation, yields our statement, hence the theorem is established.

Let us deduce the following

Corollary. *Let Z_n be a random variable on S and assume that $Z_n \to 1$ in probability. Then*

$$\lim_{n=+\infty} P(Z_n/D_n < y) = 1 - e^{-y}. \tag{13}$$

Proof. Rewrite Z_n/D_n as

$$\frac{Z_n}{D_n} = \frac{Z_n}{n}\frac{n}{D_n} = \frac{n}{D_n} + \frac{n}{D_n}\left(\frac{Z_n}{n} - 1\right).$$

If we show that the second term stochastically tends to 0, then by a well known lemma of Cramer we obtain the statement of the Corollary, namely the first term above has the limit distribution being on the right hand side of (13), as it was shown in (12). Thus we have to show that for any positive real number t,

$$P\left(\left|\frac{n}{D_n}\left(\frac{Z_n}{n} - 1\right)\right| \geqq t\right) \to 0 \quad \text{as} \quad n \to +\infty.$$

Apply that, for any fixed T,

$$P\left(\left|\frac{n}{D_n}\left(\frac{Z_n}{n} - 1\right)\right| \geqq t\right) \leqq P\left(\frac{n}{D_n} > T\right) + P\left(\left|\frac{Z_n}{n} - 1\right| \geqq \frac{t}{T}\right).$$

Here the first term is smaller than any prescribed real number by (12) and so is the second one by the assumption on Z_n, and thus the proof is complete.

As an application of the Corollary, let U_n be the number of d_k, $k \leqq n$, such that $d_k = 2$. Then from Theorem 2 we have that $2U_n \to 1$, therefore Corollary states that $2U_n/D_n$ is asymptotically exponential variate. Similar examples can be listed from Theorem 2 and from the Corollary. The occurence of the exponential distribution in these theorems suggest that the number of terms of the sequence $d_1, d_2, \ldots$ which are of the same order as D_n, follows a Poisson distribution. This is shown to hold in the next theorem.

Theorem 5. *Let Q_n denote the number of terms in the sequence $d_1, d_2, \ldots, d_n$ for which $d_j/n > y$. Then*

$$\lim_{n=+\infty} P(Q_n = k) = \frac{e^{-1/y}}{k!\, y^k}.$$

Proof. By Lemma 1 the d's are independent and identically distributed, therefore Q_n has a binomial distribution with parameters n and $1/[ny]$, the latter having been obtained in (6). Since $n/[ny] \to 1/y$, by a well known result ([1], p. 143), the limit distribution of Q_n is Poisson, as stated. The proof is terminated.

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