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ON THE CHARACTERIZATION OF WEAK CLOSURE IN HILBERT SPACE*)

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The purpose of this note is an attempt to characterize the weak closure of bounded sets in a separable Hilbert space. The problem was motivated by a recent paper of G. E. ŠILOV [3] in which he studies the Lévy-Laplace operator on a particular class of norm-closed domains in Hilbert space. In § 7 of [3] it was shown that these domains are weakly compact and that the nullspace of the Lévy-Laplacian is a dense algebra in the algebra of all weakly continuous functions on the given domain. It turns out that the (sup-norm closure of the) algebra considered by Šilov, or equivalently the algebra generated by all continuous linear functionals, on a given bounded (not weakly closed) set B is precisely that algebra of bounded weakly continuous functions which corresponds to the compactification given by the weak closure of B. It would be natural to expect this compactification of B to be "distinguished" in some way; examples suggest, however, that the weak closure may not enjoy any special properties not shared by other compactifications. We conclude the note with a few open problems.

Throughout this paper, $\mathscr{H}$ will denote an infinite dimensional separable Hilbert space over the reals, with real valued inner product. If B is a bounded subset of $\mathscr{H}$, we write B_w for the topological space $\{B, \tau\}$, where τ is the weak topology of $\mathscr{H}$ restricted to B, and $\bar{B}^w$ for the weak closure of B together with the weak topology. It is well known that $\bar{B}^w$ is compact and metrizable. If $q(x, y)$ is any admissible metric on $\bar{B}^w$ (they are all equivalent there) then $\bar{B}^w$ is isometric to the completion of B_w with respect to q. From now on let K be a closed ball in $\mathscr{H}$, containing $\bar{B}^w$, and let q be an admissible metric for the weak topology on K, with $q(x, y) \leqq 1$ for all $x, y \in K$.

If S is a set, denote by $F(S)$ the Banach algebra of all bounded (real or complex valued) functions on S, endowed with the sup-norm. If $S = T$ is a topological space, $C(T)$ will denote the subalgebra of all bounded continuous functions. If $\{X, \mathscr{U}\}$ is

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a uniform space, we denote by $\mathscr{U}(X)$ the closed subalgebra of $F(X)$ which consists of all bounded $\mathscr{U}$ – $\mathscr{V}$-uniformly continuous functions, where $\mathscr{V}$ is the usual uniformity on the space of real or complex numbers.

Definition. A real or complex valued function f defined on the bounded subset B of $\mathscr{H}$ is said to be *simple* [3] if there are (i) a finite number of vectors $a_1, \ldots, a_n$ in $\mathscr{H}$, (ii) a closed bounded domain Q in Euclidean $\boldsymbol{R}^n$, and (iii) a continuous function $\tilde{f}$ on Q such that 1° $x \in B$ implies $((a_1, x), \ldots, (a_n, x)) \in Q$ and 2° $f(x) = \tilde{f}((a_1, x), \ldots \ldots, (a_n, x))$.

The class $D(B)$ of simple functions forms an algebra (operations pointwise) – the arguments of § 1 in [3] remain valid for any bounded $B \subset \mathscr{H}$; and in the case when B is a norm-closed domain in $\mathscr{H}$, Šilov showed that $D(B)$ is equal to the nullspace of the Lévy-Laplace operator L(the latter is an unbounded operator defined in the larger algebra of "regular" functions; cf. [3]). Let $\bar{D}(B)$ denote the sup-norm closure of $D(B)$.

If $\mathscr{U}_l$ denotes the metric uniformity induced on B_w by that of $\bar{B}^w$, then $D(B) \subset \subset \mathscr{U}_l(B)$, and as $\mathscr{U}_l(B)$ is complete, also $\bar{D}(B) \subset \mathscr{U}_l(B)$. On the other hand, since the functions $\hat{a}$ given by $\hat{a}(x) = (a, x)$ $(a \in \mathscr{H})$ separate points of $\mathscr{H}$ and by the Stone-Weierstrass theorem form a system of generators for $\mathscr{U}_l(\bar{B}^w) = C(\bar{B}^w)$ – and hence also for the restriction of this algebra to B_w, namely $\mathscr{U}_l(B_w)$ – we obtain $\bar{D}(B) \supset \supset \mathscr{U}_l(B_w)$. We have proved the following.

Proposition 1. *The compactification $\bar{B}^w$ of B_w corresponds to the closed subalgebra $\mathscr{U}_l(B_w) = \bar{D}(B)$ of $C(B_w)$. This algebra is generated by the continuous linear functionals on $\mathscr{H}$.*

Proposition 2. *Let $\{X, \mathscr{U}\}$ be any uniform Hausdorff space. If $f \in \mathscr{U}(X)$ and $f^{-1} \in F(X)$, then $f^{-1} \in \mathscr{U}(X)$.*

Proof. If $f \in F(X)$, then $f^{-1} \in F(X)$ if and only if $\inf\{|f(x)| : x \in X\} = c > 0$. Suppose $f \in \mathscr{U}(X)$ and $f^{-1} \in F(X)$ now, and set $d = \sup\{|f(x)| : x \in X\}$. Then $f^{-1}(x) = (f(x))^{-1} = (r_{c,d} \circ f)(x)$ where $r_{c,d}(z) = z^{-1}$ and $r_{c,d}$ is uniformly continuous on the annulus $c \leqq |z| \leqq d$. Consequently the composition $f^{-1} = r_{c,d} \circ f$ is $\mathscr{U}$-uniformly continuous. (In $\boldsymbol{C}$ we always use the usual uniformity given by the metric $p(z_1, z_2) = |z_2 - z_1|$.)

Proposition 3. *If $f \in D(B)$ $[f \in \bar{D}(B)]$ and $f^{-1} \in F(B)$, then $f^{-1} \in D(B)$ $[f^{-1} \in \bar{D}(B)]$.*

The case of $D(B)$ was proved directly in [2]; the case for $\bar{D}(B) = \mathscr{U}_l(B_w)$ follows from the preceding propositions.

Now let $D_0(B)$ denote the set of simple functions which are representable in the form $x \mapsto g((a_1, x), \ldots, (a_m, x))$ where $a_1, \ldots, a_m \in B$, and denote by $\bar{D}_0(B)$ its sup-norm closure. Then $D_0(B)$ is an algebra, and we have the following result.

Proposition 4. *If the functions* $\hat{a}(x) = (a, x)$, $a \in B$, *separate the points of* B (*in particular, if the norm interior of* B *is not empty*) *then* $\mathscr{U}_l(B_w) = \bar{D}_0(B)$.

Let us take a look at other possible compactifications of B_w.

Lemma 1. *Let* $\{X, p\}$ *be a metric space and* a, b *two distinct points of* X. *Write* $X \setminus \{a, b\} = Y$ *and set*

$$(1) \qquad r(x, y) = \min \left[p(x, y),\, p(x, a) + p(y, b),\, p(x, b) + p(y, a)\right].$$

Then r *is a metric on* Y, *and both* r *and* p *determine the same topology on that space. Moreover* r *is strictly coarser than* p; *if* $\{X, p\}$ *is precompact, then so is* $\{Y, r\}$.

The verification is straightforward.

Proposition 5. *Let* a, b *be two distinct points of* $\bar{B}^w \setminus B_w$, $p = q$, *and* r *as defined in* (1). *Then at least one of the functions* $\hat{a}(x) = (a, x)$ *and* $\hat{b}(x) = (b, x)$ *is not* r*-uniformly continuous on* B_w, *and the compactification of* B_w *corresponding to* r *is obtained from* $\bar{B}^w$ *by identifying* a *and* b.

This is immediate from Lemma 1.

Remark. If $a \neq 0 \neq b$ in Proposition 5, then both of $\hat{a}$ and $\hat{b}$ are not r-uniformly continuous.

Lemma 2. *Let* $\{X, p\}$ *be a metric space with* $p(x, y) \leqq 1$ *for all* $x, y \in X$, *and* $a \in X$ *a non-isolated point. Set*

$$(2) \qquad d'(x, y) = \log\left(1 + \frac{4p(x, y)}{\min\left[p(x, a), p(y, a)\right]^2}\right).$$

Then d' *is a metric on* $X \setminus \{a\} = Y$ *which is strictly finer than* p, *and* $\{Y, d'\}$ *is not precompact. Moreover, the function* g *given by* $g(x) = \sin(1/p(x, a))$ *is* d'*-uniformly continuous on* Y, *but not* p*-uniformly continuous.*

Proof. A straightforward calculation shows that d' is indeed a metric on Y, and that $\{Y, d'\}$ is not precompact. We check d'-uniform continuity of g. The function sine is uniformly continuous on $\boldsymbol{R}$; so there is a function δ from $]0, 2]$ to $]0, 2\pi]$ such that $|\xi - \eta| < \delta(\varepsilon)$ implies $|\sin \xi - \sin \eta| < \varepsilon$. Then $d'(x, y) < \log(1 + 4\delta(\varepsilon))$ yields

$$\delta(\varepsilon) > \frac{p(x, y)}{\min\left[p(x, a), p(y, a)\right]^2} \geqq \frac{p(x, y)}{p(x, a)\, p(y, a)} \geqq$$

$$\geqq \frac{|p(x, a) - p(y, a)|}{p(x, a)\, p(y, a)} = \left|\frac{1}{p(x, a)} - \frac{1}{p(y, a)}\right|$$

and hence

$$\left|\sin \frac{1}{p(x, a)} - \sin \frac{1}{p(y, a)}\right| < \varepsilon .$$

Thus g is d'-uniformly continuous, but it clearly is not p-uniformly continuous.

Lemma 3. *Let $\{X, d\}$ be a metric space which is not precompact, and $d(x, y) \leqq 1$ for all $x, y \in X$. Let $\mathscr{D}^*$ be the coarsest uniformity on X which leaves all bounded d-uniformly continuous functions still $\mathscr{D}^*$-uniformly continuous. Then $\mathscr{D}^*$ is also the finest uniformity on X which* (i) *is Hausdorff,* (ii) *determines the same topology as d, and* (iii) *makes $\{X, \mathscr{D}^*\}$ precompact. The uniformity $\mathscr{D}^*$ is generated by the family of semi-metrics $\hat{b}(x, y) = |d(b, x) - d(b, y)|$, $b \in X$ (the $\hat{b}$ form a subbase). If X is separable in the topology given by d, and $\{b_n : n \in N\}$ is a countable dense set in X, then $\mathscr{D}^*$ is determined by the metric*

$$d^*(x, y) = \sum_{n \in N} 2^{-n} \hat{b}_n(x, y) .$$

Proof. The first part is an exercise in § 1 of [1]. The verification that $\mathscr{D}^*$ is generated by the $\hat{b}$, or by d^*, respectively, is left to the reader.

Proposition 6. *Let $a \in \bar{B}^w \setminus B_w$. Then there exists a metric r which is strictly finer than q on B but which determines the same topology as does q (namely the weak topology on B) such that $\{B, r\}$ is precompact, and the function $g(x) = \sin(1/q(x, a))$ is r-uniformly continuous.*

Proof. We give two examples of such a metric. (i) Let $X = \bar{B}^w$ and $p = q$ in Lemma 2 (a is not isolated in $\bar{B}^w$), and $d = \min\{d', 1\}$ in Lemma 3. Then $r = d^*$ has the desired properties. (ii) Let f be any bounded continuous function on B_w which is not q-uniformly continuous, for instance $f(x) = \sin(1/q(x, a))$. Then $r(x, y) = q(x, y) + |f(x) - f(y)|$ is a metric on B for which $\{B, r\}$ is precompact, r determines the weak topology on B, and f is r-uniformly continuous.

The above results are summarized as follows.

Theorem. *The weak closure $\tilde{B}$ of a bounded set B in a separable, real Hilbert space $\mathscr{H}$ is the compactification determined by the algebra generated by all continuous functionals on $\mathscr{H}$ (or equivalently, by the simple functions). If B is not weakly closed, it has metrizable compactifications which are "larger" than $\tilde{B}$, and if $\tilde{B} \setminus B$ contains at least two points, $\tilde{B}$ also has metrizable compactifications which are "smaller" than B.*

We conclude with some unanswered questions.

Problem 1. (a) Characterize all metrics on B which determine the weak topology, in particular those arising from compactifications. (b) Give an example of a bounded set B in $\mathscr{H}$ which is not weakly closed, of a nonmetrizable uniformity $\mathscr{U}$ on B determining the weak topology – perhaps so that $\{B, \mathscr{U}\}$ is precompact, and of a $\mathscr{U}$-uniformly continuous function which is not q-uniformly continuous.

Problem 2. Does every $\mathscr{U}_l$-uniformly continuous function on B have a $\mathscr{U}_l$-uniformly continuous extension to the closed convex hull of B (perhaps even to K)?

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