Héctor J. Sussmann

The control problem $\dot{x} = (A(1-u) + Bu)x$: A comment on an article by J. Kučera

Czechoslovak Mathematical Journal, Vol. 22 (1972), No. 3, 423–426

Persistent URL: http://dml.cz/dmlcz/101111

Czechoslovak Mathematical Journal, 22 (1972), Praha

THE CONTROL PROBLEM $\dot{x} = (\mathrm{A}(1 - u) + \mathrm{B}u)x$: A COMMENT ON AN ARTICLE BY J. KUČERA

HÉCTOR J. SUSSMANN, Chicago*)

(Received March 12, 1971)

In an article recently published in this journal ([1]), J. KUČERA studied the control problem $\dot{x} = (\mathrm{A}(1 - u) + \mathrm{B}u)\,x$. The main results of [1] are that the set $\mathscr{A}(\omega, T)$ of points attainable at time $T > 0$ from a fixed point ω is an "integral manifold of the distribution $\mathscr{B}(\mathrm{A}, \mathrm{B})$" ([1], Theorem 2.2), and that the set $\mathscr{A}'(\omega, T) = \bigcup\{\mathscr{A}(\omega, T) : : 0 \leqq t \leqq T\}$ is an "integral manifold of the distribution $\mathscr{U}(\mathrm{A}, \mathrm{B})$" ([1], Theorem 2.1). The purpose of this note is to show that Lemma 2.8 [1], which is a fundamental step in the proof of Theorems 2.1 and 2.2, is false. The natural question to be asked now is whether these results are nevertheless valid; it will be shown in a forthcoming paper that they are. The proof, however, is based on a completely different technique.

We quote the statement of Lemma 2.8 of [1]:

"[Let] $T > 0$, $\delta \in (0, \frac{1}{2})$, $u \in M(\delta, 1 - \delta)$. *Let the function* u *be not constant in* $\langle 0, T\rangle$ (*not equivalent with a constant function*), *then*

$$\mathscr{V}(x(T, u)) \subset \bigcup_{r=1}^{\infty} r \,.\, K_u(T)\,."$$

The notations of the above statement have the following meaning:

a) "(α, β)" (resp. $\langle\alpha, \beta\rangle$) is the open (resp. closed) interval with endpoints α, β.

b) "$M(\alpha, \beta)$" is the set of all measurable functions in $(0, \infty)$ whose values lie in $\langle\alpha, \beta\rangle$, $\alpha < \beta$.

c) $t \to x(t, u)$ is the solution of the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\, x(t) = (\mathrm{A}(1 - u(t)) + \mathrm{B}u(t)\,)\, x(t)$$

which satisfies $x(0) = \omega$. Here ω is a fixed element of $\boldsymbol{R}^n$ (n-dimensional real space), and A, B are fixed elements of $\mathscr{M}_n$ (the set of all n by n real matrices).

*) This work was performed while the author was at the Division of Engineering and Applied Physics, Harvard University, Cambridge, Mass., U.S.A.

d) "$\mathscr{V}$" is the "distribution created by $\mathscr{B}$", i.e. the mapping that assigns to each $x \in \mathbf{R}^n$ the set $\mathscr{V}(x)$ of all elements of $\mathbf{R}^n$ of the form Px, $P \in \mathscr{B}$.

e) "$\mathscr{B}$", or "$\mathscr{B}(\mathrm{A}, \mathrm{B})$" is "the smallest linear space of n by n matrices which contains the matrix $\mathrm{C}(=\mathrm{B} - \mathrm{A})$ and, with each $\mathrm{P} \in \mathscr{B}$, contains also both matrices $[\mathrm{A}, \mathrm{P}]$ and $[\mathrm{B}, \mathrm{P}]$" ([1], Def. 2.2; the notation "$[\mathrm{M}, \mathrm{N}]$" means "NM − MN").

f) $K_u(T)$ is the set of all vectors $x_1(T, v)$, $v \in M(-1, 1)$, where

$$x_1(T, v) = \mathrm{X}(T)\left(\left(\int_0^T \mathrm{X}^{-1}(t)\, \mathrm{C}\mathrm{X}(t)\, v(t)\, \mathrm{d}t\right)\omega\right.,$$

and where

g) $t \to \mathrm{X}(t)$ is the n by n matrix-valued solution of

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{X}(t) = (\mathrm{A}(1 - u(t)) + \mathrm{B}u(t))\,\mathrm{X}(t)$$

which satisfies $\mathrm{X}(0) = \mathrm{I}$, (I is the n by n identiy matrix).

We shall show that Lemma 2.8 is false by means of a counterexample. Consider the space $\mathscr{P}_0$ of all real polynomials in two *noncommuting* variables y, z. Form the space $\mathscr{P}$ by equating to zero all the monomials of degree 5 or more (in other words $\mathscr{P}_0$ is the free algebra over the reals generated by y and z, and $\mathscr{P}$ is the quotient of $\mathscr{P}_0$ by the ideal generated by all the monomials of degree 5). Thus, $\mathscr{P}$ is a 31-dimensional real vector space, and the monomials $1, y, z, y^2, yz, zy, z^2, y^3, y^2z, yzy, zy^2, yz^2, zyz, z^2y, z^3, y^4, y^3z, y^2zy, yzy^2, zy^3, y^2z^2, yzyz, zy^2z, yz^2y, zyzy, z^2y^2, yz^3, zyz^2, z^2yz, z^3y, z^4$ are a basis for $\mathscr{P}$. Moreover, $\mathscr{P}$ is an associative algebra over the reals, with the obvious multiplication table (for instance: $zy \,.\, zy = zyzy$, $zy \,.\, y^3 = 0$, etc.).

By means of this basis we can (and shall) identify $\mathscr{P}$ with $\mathbf{R}^{31}$. In $\mathscr{P}$, the mappings $p(y, z) \to yp(y, z)$ and $p(y, z) \to zp(y, z)$ are linear. Via the above mentioned identification, we obtain two 31 by 31 matrices M_y and M_z such that these mappings correspond to $x \to \mathrm{M}_y x$ and $x \to M_z x$, respectively. We let $\mathrm{A} = \mathrm{M}_y$, $\mathrm{B} = \mathrm{M}_z + \mathrm{M}_y$, so that $\mathrm{C} = \mathrm{M}_z$. To begin with, we compute the space $\mathscr{B}$. It is clear from the definition that $\mathscr{B}$ is the smallest linear space that contains C such that, if $\mathrm{P} \in \mathscr{B}$, then $[\mathrm{A}, \mathrm{P}]$ and $[\mathrm{C}, \mathrm{P}]$ belong to $\mathscr{B}$. Thus $\mathscr{B}$ is the linear hull of the set $\mathscr{E}$ of all matrices $[\mathrm{Q}_1, [\mathrm{Q}_2, \ldots, [\mathrm{Q}_{k-1}, \mathrm{Q}_k] \ldots]]$, where k is an integer >0, and where $\mathrm{Q}_i = \mathrm{A}$ or $\mathrm{Q}_i = \mathrm{C}$ for $i = 1, \ldots, k - 1$, $\mathrm{Q}_k = C$. Using the facts that $[\mathrm{C}, \mathrm{C}] = 0$ and that $[\mathrm{A}, [\mathrm{C}, [\mathrm{A}, \mathrm{C}]]] = [\mathrm{C}, [\mathrm{A}, [\mathrm{A}, \mathrm{C}]]]$ (an immediate consequence of the equality $[\mathrm{P}, \mathrm{P}] = 0$ and of the Jacobi identity $[\mathrm{P}, [\mathrm{Q}, \mathrm{R}]] = [[\mathrm{P}, \mathrm{Q}], \mathrm{R}] + [\mathrm{Q}, [\mathrm{P}, \mathrm{R}]]$) we see that the following are *all* the elements of $\mathscr{E}$ corresponding to $k \leqq 4$:

$$\mathrm{M}_1 = \mathrm{C}\,,\quad \mathrm{M}_2 = [\mathrm{A}, \mathrm{C}]\,,\quad \mathrm{M}_3 = [\mathrm{A}, [\mathrm{A}, \mathrm{C}]]\,,\quad \mathrm{M}_4 = [\mathrm{C}, [\mathrm{A}, \mathrm{C}]]\,,$$
$$\mathrm{M}_5 = [\mathrm{A}, [\mathrm{A}, [\mathrm{A}, \mathrm{C}]]],\quad \mathrm{M}_6 = [\mathrm{C}, [\mathrm{A}, [\mathrm{A}, \mathrm{C}]]] \quad \text{and} \quad \mathrm{M}_7 = [\mathrm{C}, [\mathrm{C}, [\mathrm{A}, \mathrm{C}]]]\,.$$

In addition, all the elements of $\mathscr{E}$ corresponding to $k \geqq 5$ vanish. This is so because, via our identification of $\mathbf{R}^{31}$ with $\mathscr{P}$ (and of the corresponding identification of $\mathscr{M}_{31}$

with the set of endomorphisms of the vector space $\mathscr{P}$), every element $Q = [Q_1, [Q_2, \ldots \ldots, [Q_{k-1}, Q_k] \ldots]]$ of $\mathscr{E}$ corresponds to the multiplication in $\mathscr{P}$ by a homogeneous polynomial $p(Q)$ of degree k (for instance, if $Q = [A, C]$, then $p(Q) = zy - yz$). Since every homogeneous polynomial of degree $\geqq 5$ vanishes in $\mathscr{P}$, our assertion follows.

Thus $\mathscr{B}$ is the linear hull of $M_1, \ldots, M_7$. We show that these matrices are linearly independent. It is sufficient to prove that the multiplications by the corresponding polynomials $p(M_1), \ldots, p(M_7)$ are linearly independent. If these multiplications were not independent, then the images of the polynomial 1 would be dependent, i.e. the polynomials $p(M_1), \ldots, p(M_7)$ would be dependent. Thus, it is sufficient to show that these polynomials are independent. But $p(M_1), \ldots, p(M_7)$ are homogeneous polynomials both in y and z, and no two of them have the same degrees both in y and z. Hence they are independent.

We have shown that $M_1, \ldots, M_7$ form a basis for $\mathscr{B}$, so that *$\mathscr{B}$ has dimension seven.*

We shall take ω to be the element of $\mathbf{R}^{31}$ which corresponds to the polynomial 1 of $\mathscr{P}$. Thus, $\mathscr{V}(\omega)$ is the linear hull of $p(M_1), \ldots, p(M_7)$, and $\dim \mathscr{V}(\omega) = 7$. We take δ to be an arbitrary element of $(0, \frac{1}{2})$, and define the control u by $u(t) = \delta + t$ for $0 \leqq t \leqq T$, where $T = 1 - 2\delta$. Thus all the assumptions of Lemma 2.8 of [1] hold. We show that the dimension of $\mathscr{V}(x(T, u))$ is also seven. This is an immediate consequence of Lemma 2.11 of [1], or it can be proved directly as follows: the equation

$$\frac{d}{dt} x(t, u) = (A + u(t)\, C)\, x(t, u)$$

implies that the derivative of the polynomial $x(t, u)$ is a polynomial in y, z without a constant term. This implies that the constant term of $x(t, u)$ is 1 for all t (because $x(0, u) = \omega = 1$). From this it follows immediately that the seven polynomials $p(M_i)\, x(t, u)$ are linearly independent.

We shall show that the dimension of the subspace $\bigcup_{r=1}^{\infty} r \,.\, K_u(T)$ is not greater than *six*. To begin with, $\bigcup_{r=1}^{\infty} r \,.\, K_u(T)$ is obviously the set of all elements of the form

$$X(T) \left(\int_0^T X^{-1}(t)\, CX(t)\, v(t)\, dt \right) \omega\,,$$

where v is an arbitrary bounded measurable function in $\langle 0, T \rangle$. We see immediately that this is the same as the linear hull L'' of $X(T)\, X^{-1}(t)\, CX(t)\, \omega$, $t \in \langle 0, T \rangle$. The dimension of L'' is the same as that of the linear hull L' of all the elements $X^{-1}(t)\,.\,CX(t)\, \omega$ (because $X(T)$ is nonsingular). Finally, this dimension is not greater than that of the linear hull L of all the matrices $X^{-1}(t)\, CX(t)$, $t \in \langle 0, T \rangle$.

Thus, it is sufficient to show that $\dim L \leqq 6$. Since $u(t) = t + \delta$, the function $t \to X^{-1}(t)\, CX(t) = Y(t)$ is analytic. Thus, L is the linear hull of the coefficients of the power series expansion of Y in a neighborhood of $t = 0$ or, equivalently, L is the linear hull of $\{(\mathrm{d}^n/\mathrm{d}t^n)\, Y(t)|_{t=0} : n = 0, 1, \ldots\}$.

Since $(\mathrm{d}/\mathrm{d}t)\, \mathrm{X}(t) = (\mathrm{A} + u(t)\, \mathrm{C})\, \mathrm{X}(t)$, we see that

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{X}^{-1}(t) = -\mathrm{X}^{-1}(\mathrm{A} + u(t)\, \mathrm{C})\,.$$

If $\mathrm{M}(t)$ is any matrix-valued function, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{X}^{-1}(t)\, \mathrm{M}(t)\, \mathrm{X}(t)) = -\mathrm{X}^{-1}(t)\,(\mathrm{A} + u(t)\, \mathrm{C})\, \mathrm{M}(t)\, \mathrm{X}(t) +$$

$$+ \mathrm{X}^{-1}(t)\, \mathrm{M}(t)\,(\mathrm{A} + u(t)\, \mathrm{C})\, \mathrm{X}(t) + \mathrm{X}^{-1}(t) \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{M}(t)\,.\, \mathrm{X}(t)\,.$$

Applying this formula to the successive derivatives of Y we obtain

$$\frac{\mathrm{d}^i}{\mathrm{d}t^i} \mathrm{Y}(t) = \mathrm{X}^{-1}(t)\, \mathrm{F}_i(t)\, \mathrm{X}(t)\,,$$

where

$$\mathrm{F}_0(t) \equiv \mathrm{C}\,, \quad \mathrm{F}_1(t) \equiv [\mathrm{A}, \mathrm{C}]\,, \quad \mathrm{F}_2(t) = [\mathrm{A} + (t + \delta)\, \mathrm{C}, [\mathrm{A}, \mathrm{C}]]\,,$$
$$\mathrm{F}_3(\mathrm{t}) = [\mathrm{A} + (t + \delta)\, \mathrm{C}, [\mathrm{A} + (t + \delta)\, \mathrm{C}, [\mathrm{A}, \mathrm{C}]]] + [\mathrm{C}, [\mathrm{A}, \mathrm{C}]]\,,$$
$$\mathrm{F}_4(t) = [\mathrm{C}, [\mathrm{A} + (t + \delta)\, \mathrm{C}, [\mathrm{A}, \mathrm{C}]]] + 2[\mathrm{A} + (t + \delta)\, \mathrm{C}, [\mathrm{C}, [\mathrm{A}, \mathrm{C}]]]\,,$$
$$\mathrm{F}_5(t) \equiv 3[\mathrm{C}, [\mathrm{C}, [\mathrm{A}, \mathrm{C}]]] \quad \text{and} \quad \mathrm{F}_6(t) \equiv 0\,.$$

In the above computations we have used the fact that every element of $\mathscr{E}$ corresponding to $k \geqq 5$ vanishes.

Our computations show that, of all the matrices $(\mathrm{d}^n/\mathrm{d}t^n)\, \mathrm{Y}(t)|_{t=0}$, only those for $n = 0, \ldots, 5$ are nonzero. Thus, the dimension of L is not greater than 6.

The preceeding remarks show that, in our example, the dimension of $\mathscr{V}(\mathrm{X}(T, u))$ is seven, while the dimension of $\bigcup_{r=1}^{\infty} r\,.\, K_u(T)$ is less than six. Therefore, the conclusion of Lemma 2.8 of [1] does not hold, even though all the assumptions are satisfied. Thus, Lemma 2.8 of [1] is false.

Reference

[1] *J. Kučera:* Solution in large of Control Problem $\dot{x} = (\mathrm{A}(1 - u) + \mathrm{B}u)\, x$. Czech. Math. J. *16* (*91*), 1966, 600–623.

Author's address: Department of Mathematics, University of Chicago, Chicago, Illinois 60637, U.S.A.