N. P. Mukherjee
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QUASICOMMUTATIVE SEMIGROUPS I

N. P. MUKHERJEE, Morgantown

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The kern of a group G is the totality of all elements in G each of which normalises every subgroup of G and it is well known that the kern is a subgroup which is either abelian or Hamiltonian.

Following the idea of the kern we introduce here the idea of quasicommutativity and investigate the structure of quasicommutative semigroups.

Definition. An element a of a semigroup S is *quasicommutative* if and only if for every element b in S, $ab = b^r a$ holds for some positive integer $r \geqq 1$.

Remark. It should be noted that r may not be same for different choices of b.

Definition. A semigroup S is called *quasicommutative* if every element of S is quasicommutative.

Remarks. 1. The totality of all quasicommutative elements of a semigroup S is a subsemigroup of S.

2. If a is a quasicommutative element of a semigroup S and e is any idempotent of S then $ea = ae$.

3. In a quasicommutative semigroup all the idempotents commute with each other.

Proposition. *If for any two elements a, b of a semigroup S $ab = b^r a$ for some positive integer r then for every positive integer m the following are true:* (1) $ab^m = b^{mr}a$. (2) $a^m b = b^{r^m} a^m$. (3) $(ab)^m = b^{r+r^2+\ldots+r^m} a^m$. (4) $(ba)^m = b^{1+r+r^2+\ldots+r^{m-1}} a^m$.

Proof. (1) The result is true for $m = 1$ and assume it true for $m = n$, i.e. $ab^n = b^{nr}a$. Then $ab^{n+1} = b^{nr}ab = b^{nr}b^r a = b^{(n+1)r}a$. Hence it follows that $ab^m = b^{mr}a$ holds for all positive integers m.

(2) The result is true for $m = 1$ and assume it true for $m = n$, i.e. $a^n b = b^{r^n} a^n$. Then $a^{n+1}b = ab^{r^n}a^n = b^{r \cdot r^n} a \,.\, a^n$ by (1). Hence $a^{n+1}b = b^{r^{n+1}} a^{n+1}$ and the result follows by induction.

(3) The result is true for $m = 1$ and assume it true for $m = n$, i.e. $(ab)^n = b^{r+r^2+\dots+r^n}a^n$. Hence $(ab)^{n+1} = b^{r+r^2+\dots+r^n}a^n ab = b^{r+r^2+\dots+r^n}a^{n+1}b = b^{r+r^2+\dots+r^{n+1}}a^{n+1}$ by (2). The result therefore follows by induction.

(4) $(ba)^2 = ba \cdot ba = b \cdot b^r a^2 = b^{1+r}a^2$ and therefore the result is true for $m = 2$. Assume now that the result holds for $m = n$. Then, $(ba)^n = b^{1+r+\dots+r^{n-1}}a^n$ and therefore $(ba)^{n+1} = b^{1+r+\dots+r^{n-1}}a^n ba = b^{1+r+\dots+r^n}a^{n+1}$ by (2). Hence the result follows by induction.

Theorem 1. *A periodic quasicommutative semigroup S is a union of a semilattice Ω of quasicommutative semigroups.*

Proof. If $a \in S$ then $\langle a \rangle$ has exactly one idempotent e and suppose $S_e = \{x \mid x^n = e$, for some positive integer $n\}$. Note that $S_e \cap S_f = \emptyset$ and so S is the disjoint union of all S_e, $e \in \Omega$ = set of idempotents in S. Suppose $a, b \in S_e$ and let $ab = b^r a$. From the previous lemma, $(ab)^n = b^{r+r^2+\dots+r^n}a^n = b^{r+r^2+\dots+r^n}e$ and therefore $(ab)^{n^2} = b^{n(r+r^2+\dots+r^n)}e = e \cdot e = e$. Thus $ab \in S_e$ and S_e is a semigroup. Also, if $x \in S_e$, $y \in S_f$, $x^n = e$, $y^m = f$ and $xy = y^t x$ then, $(xy)^n = y^{t+t^2+\dots+t^n}x^n = y^{t+t^2+\dots+t^n}e$. Hence, $(xy)^{mn} = y^{m(t+t^2+\dots+t^n)}e = fe = ef$ and $S_e S_f \subseteq S_{ef}$ and the theorem follows.

Theorem 2. *A regular quasicommutative semigroup S is a union of groups.*

Proof. Let a be an element of a regular quasicommutative semigroup S and Ω be its set of idempotents. Suppose $axa = a$, $xax = x$ and $ax = e$. Then $a = axa = ea = ae$ and $xe = ex = x$. Now, if H_e is the maximal group of units in eSe then note that $a \in eSe$ and $x \in eSe$. Thus, if $xa = e_1$ then $ee_1 = e_1 e = e_1$ and by similar arguments we can show that $e_1 S e_1$ contains $ax = e$ and $ee_1 = e_1 e = e$. Hence $e = e_1$ and therefore $a \in H_e$. S is therefore a disjoint union of the maximal subgroups H_e and the theorem is proved.

Remarks. 1. The structure of H_e is well known. If all elements of H_e are of finite order then it is Hamiltonian. A Hamiltonian group is a noncommutative group in which all subgroups are normal. It is the direct product of a quaternion group, a commutative group in which the elements are of order 2 and a commutative group in which the elements are of odd order. If however H_e contains an element of infinite order then it is abelian.

2. Since the structure of quasicommutative groups is completely known it may be seen without any difficulty that the quasicommutative regular semigroups can be constructed in the manner described in theorem 4.11 in [2].

3. Since the idempotents in a quasicommutative semigroup commute with each other it follows that a regular quasicommutative semigroup is an inverse semigroup.

We shall now generalise a result about commutative semigroups for semigroups which are quasicommutative. Suppose η is a relation defined in a quasicommutative semigroup as follows: $a\eta b$ if and only if $ax = b^m$, $by = a^n$ where $x, y \in S$ and m, n are positive integers. Synonymously, we say $a\eta b$ if and only if $a \mid b^m$ and $b \mid a^n$.

Theorem 3. *The relation defined above in a quasicommutative semigroup S is a congruence and S/η is a maximal semilattice homomorphic image of S.*

Proof. Evidently, η is reflexive and symmetric. Suppose now $a\eta b$ and $b\eta c$. These imply $ax = b^m$ and $by = c^n$. Since S is quasicommutative, we have $by = y^k b$ and therefore by the lemma we proved earlier, $(by)^m = y^{k+k^2+\ldots+k^m} b^m$. Hence $y^{k+k^2+\ldots+k^m} ax = c^{mn}$, i.e. $a^v(y^{k+k^2+\ldots+k^m}x) = c^{mn}$, i.e. $a(a^{v-1}y^{k+k^2+\ldots+k^m}x) = c^{mn}$. Thus it follows that a divides a power of c and since η is symmetric we may repeat the above arguments to show that c also divides a power of a. Hence η is transitive.

We now show that η is right as well as left compatible. Let d be an element of S and suppose $bd = d^r b$ so that $(bd)^m = d^{r+r^2+\ldots+r^m} b^m = d^{r+r^2+\ldots+r^m} ax = a^w d^{r+r^2+\ldots+r^m} x = a(a^{w-1} d^{r+r^2+\ldots+r^m}) x = a(d^{s(r+r^2+\ldots+r^m)} a^{w-1}) x = ady_0$. Hence $ad \mid (bd)^m$. But since b divides a power of a and S is quasicommutative, we can repeat the same arguments to show bd divides a power of ad. Thus η is right compatible. We now show that η is left compatible. We shall first show that db divides a power of da. Let $da = a^u d$. Then $(da)^h = a^{u+u^2+\ldots+u^h} d^h = a^{u_0} d^h = a^{n_0} a^{u_0 - n_0} d^h$ since b divides a power of a and $b\bar{x} = a^{n_0}$ and h is large enough so that $u_0 > n$. Hence $(da)^h = d^{hh_1} b\bar{x} a^{u_0 - n_0} = d(d^{hh_1 - 1} b)\, \bar{x} a^{u_0 - n_0} = d(b^{h_2} d^{hh_1 - 1})\, \bar{x} a^{u_0 - n_0} = db\bar{\bar{x}}$. Hence db divides a power of da. By similar arguments we can show that da divides a power of db. Thus η is left compatible and so is a congruence. Evidently, for any two arbitrary elements a, b in S, $a\eta a^2$ and $a\eta b\eta = (ab)\,\eta$. But $ab = b^r a$ and therefore $(ab)\,\eta = (b^r a)\,\eta = b\eta a\eta = (ba)\,\eta$. Thus $a\eta \,.\, b\eta = b\eta \,.\, a\eta$ and S/η is a semilattice.

Suppose now ϱ is any idempotent congruence on S and $a\eta b$. This implies $ax = b^m$ and $by = a^n$ and therefore, $ax\varrho ax \varrho b^m \varrho b$. Similarly, $by\varrho a$. But $b\varrho b^2$ implies by $\varrho b^2 y$ and since $b^2 y \varrho ba$ it follows that $a\varrho ba$. Also, $a\varrho a^k$ for each integer $k \geqq 1$ and so if $ba = a^s b$ it follows that $ab\varrho a^s b \varrho ba$. Hence $a\varrho ba \varrho ab$. But $ax\varrho b$ implies $a^2 x \varrho ab$ and since $a\varrho a^2$ implies $ax\varrho a^2 x$ it follows that $ab\varrho b$. Thus $a\varrho b$ and therefore, $\eta \subseteq \varrho$ and S/η is the maximal semilattice homomorphic image.

Remark. For any two elements a, b in S, $a^0 b$ and $b^0 a$ should be interpreted as b and a respectively.

Theorem 4. *Every quasicommutative semigroup S is uniquely expressible as a semilattice Y of Archimedian semigroups S_α, $\alpha \in Y$. The semilattice Y is isomorphic with the maximal semilattice homomorphic image S/η of S and each S_α is an equivalence class of S* mod *η.*

Proof. Let S be a quasicommutative semigroup and η be the relation as before. From the previous theorem we know that S/η is a semilattice. Let A be one of the equivalence classes and if $a, b \in A$ then $a\eta b$ implies $ab\eta b^2 \eta b$. Thus $ab \in A$ and A is a semigroup. We now show that A is Archimedian.

$a\eta b$ implies $ax = b^m$, $by = a^n$ and suppose $xb = b^s x$. Distinguish two cases. Case I. $s = 1$. In this case $xb = bx$ and so $axb = b^{m+1}$, i.e. $abx = b^{m+1}$. Hence

$(bx)^u a = b^{m+1}$ for some positive integer u which implies bx divides a power of b and since b divides bx it follows that $bx\eta b$, i.e. $bx \in A$.

Case II. $s > 1$. From $axb = b^{m+1}$ we get $a(b^s x) = b^{m+1}$ and therefore $ab(b^{s-1}x) = b^{m+1}$, i.e. $(b^{s-1}x)^i\, ab = b^{m+1}$ for some positive integer i. This implies that $b^{s-1}x$ divides a power of b. Also, $b^{s-1}x \,.\, b^{s-1}x = b^{s-1}b^{s(s-1)}x^2 = b^{s^2-1}x^2$ and therefore $b \,.\, b^{s^2-2}x^2 = (b^{s-1}x)^2$. Hence b also divides a power of $b^{s-1}x$ so that $b^{s-1}x\eta b$. This implies $b^s x\eta b^2\eta b$, i.e. $xb\eta b$. Thus $xb \in A$. From $by = a^n$ we get $bya = a^{n+1}$ and since $ya = a^h y$ for some positive integer h it follows that by using the same arguments we can show that $ya \in A$. Hence it is evident that A is Archimedian.

Suppose now S is a semilattice Y of Archimedian semigroups S_α, $\alpha \in Y$. If $a, b \in S_\alpha$ then clearly $a\eta b$. Also, if $a\eta b$ and $a \in S_\alpha$, $b \in S_\beta$ then from $ax = b^m$, $by = a^n$ we get $ax \in S_{\alpha\gamma}$ and $b^m \in S_\beta$ where $x \in S_\gamma$. Hence $\alpha\gamma = \beta$ implies $\alpha\beta = \beta$ i.e. $\beta \leqq \alpha$. Similarly $\alpha \geqq \beta$ and therefore $\alpha = \beta$ and $a, b \in S_\alpha$. The theorem is proved completely.

Next, we define a relation σ on a quasicommutative semigroup as follows. $a\sigma b$ if and only if $ab^n = b^{n+1}$ and $ba = a^{n+1}$ for some positive integer n. If $ab^n = b^{n+1}$ and $ba^m = a^{m+1}$ and say $m > n$ then we can multiply the first equation by b^{m-n} and get $ab^m = b^{m+1}$. Thus $ab^n = b^{n+1}$ and $ba^m = a^{m+1}$ may be said to describe the relation equivalently.

Theorem 5. *If S is a quasicommutative semigroup than σ is a congruence on S.*

Proof. It is evident from the definition of σ that it is reflexive and symmetric. We shall now show that is transitive. Suppose $a\sigma b$ and $b\sigma c$. These imply, $ab^n = b^{n+1}$ and $ba^n = a^{n+1}$ and also, $bc^m = c^{m+1}$, $cb^m = b^{m+1}$. From the second equation we get $b^i c^m = c^{m+i}$ for every positive integer i so that $ab^n c^m = b^{n+1}c^m = c^{m+n+1}$, i.e. $ac^{m+n} = c^{m+n+1}$. Similarly one can obtain $ca^{m+n} = a^{m+n+1}$ from the first equation. Therefore, $a\sigma c$. To show σ is a congruence we need to show that σ is both right and left compatible.

Suppose $a\sigma b$ and $c \in S$. We first show that $ac\sigma bc$. Let $cb = b^t c$ for some positive integer t. This implies $ac \,.\, (bc)^k = ac \,.\, b^{1+t+t^2+\ldots+t^{k-1}}c^k = ab^{t+t^2+t^3+\ldots+t^k}c^{k+1}$. In view of the equality $ab^n = b^{n+1}$ we have $ab^r = b^{r+1}$ for all $r \geqq n$ and choosing k large enough (so that $t + t^2 + \ldots + t^k \geqq n$) we get $ac \,.\, (bc)^k = b^{1+t+t^2+\ldots+t^k}c^{k+1} = (bc)^{k+1}$. Again, suppose $ca = a^u c$ and choosing h large enough we get by the same arguments as before $bc \,.\, (ac)^h = (ac)^{h+1}$. Thus $ac\sigma bc$. Similarly $ca\sigma cb$. Hence it follows that σ is a congruence.

Theorem 6. *In every quasicommutative semigroup S the relation σ is separative.*

Proof. Let a and b be two elements of S such that $ab\sigma a^2\sigma b^2$. Since S is quasicommutative we have $ba = a^k b$ for some positive integer k. If $k = 1$ then $ab = ba$ and $ab\sigma a^2\sigma b^2$ implies $ab \,.\, a^{2m} = a^{2m+2}$, i.e. $ba^{2m+1} = a^{2m+2}$ and $ab \,.\, b^{2m} = b^{2m+2}$, i.e. $ab^{2m+1} = b^{2m+2}$. Thus $a\sigma b$. Now suppose $k > 1$. We have $a^k b\sigma a^{k+1}$ (since σ is a congruence) which implies $ba\sigma a^{k+1}$ and therefore $ba^k\sigma a^{2k}$. Hence, $ba^k(a^{2k})^n =$

$= (a^{2k})^{n+1}$, i.e. $b(a^k)^{2n+1} = (a^k)^{2n+2}$ for some positive integer n. Again, $a^k b = ba$ implies $a^k b^{2m+2} = bab^{2m+1} = b^{2m+3}$. Thus we may conclude that $b \sigma a^k$ and this implies $b^2 \sigma a^k b = ba$ i.e. $b^2 \sigma ba$. Hence $ba \sigma a^2$ and for some positive integer r, we have $ba \,.\, a^{2r} = a^{2r+2}$ i.e. $ba^{2r+1} = a^{2r+2}$ and together with $ab^{2m+1} = b^{2m+2}$ it follows that $a \sigma b$.

It is conjectured that S/σ is the maximal separative homomorphic image of S. It may not be out of place to make a remark about the characters of quasicommutative semigroups which we hope to investigate in another paper. WARNE and WILLIAMS in [3] extended some results of SCHWARZ on finite commutative semigroups to infinite abelian inverse semigroups. It can be seen without difficulty that most of the results of Warne and Williams can be extended to semigroups which are quasi-commutative.

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Bibliography

[1] *R. Baer,* Der Kern, eine characteristische Untergruppe, Composito Mathematica, vol. *1* (1935), pp. 254—283.

[2] *A. H. Clifford* and *G. B. Preston,* The algebraic theory of semigroups, vol. *1* (1961), American Mathematical Society.

[3] *R. J. Warne* and *L. K. Williams,* Characters on inverse semigroups, Czechoslovak Math. Journal, vol. *11* (1961), pp. 150—155.

Author's address: West Virginia University, Morgantown, West Virginia 26506.