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THE CONTROL PROBLEM $\dot{x} = A(u)\,x$

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In this article we shall study the structure of the attainable sets corresponding to the control problem

$$(1) \qquad \dot{x} = A(u)\,x$$

where

(a) x is a (column) vector in n-dimensional space $\mathbf{R}^n$,

(b) $u = (u_1, \ldots, u_m)$, where $0 \leqq u_i \leqq 1$ for $i = 1, \ldots, m$,

and

(c) $A(u)$ is an $n \times n$ matrix-valued polynomial in the u_i's.

For each multiindex $\alpha = (i_1, \ldots, i_m)$, where the i_j are nonnegative integers, we let

$$(2) \qquad u^\alpha = u_1^{i_1} \,.\, u_2^{i_2} \ldots u_m^{i_m} \,.$$

With this notation, assumption (c) can be restated as follows: we have

$$(3) \qquad A(u) = \textstyle\sum_\alpha u^\alpha A_\alpha \,,$$

where the A_α are constant $n \times n$ matrices, and where $A_\alpha = 0$ except for finitely many multiindices α.

We will associate with the system (1) a family $\mathbf{F}$ of connected submanifolds of $\mathbf{R}^n$ with the property that through every $x \in \mathbf{R}^n$ there passes a unique submanifold $\mathbf{F}(x)$ belonging to $\mathbf{F}$. The members of $\mathbf{F}$ will be defined as the maximal integral manifolds of a certain involutive family $\mathbf{D}$ of vector fields. Let $\mathbf{A}(x, t)$ denote, for $x \in \mathbf{R}^n$, $t > 0$, the set of all points that can be reached from x in no more than t units of time. We will prove:

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Theorem 1. *The set* $\mathbf{A}(x, t)$ *is contained in* $\mathbf{F}(x)$. *Moreover, in the topology of* $\mathbf{F}(x)$, $\mathbf{A}(x, t)$ *is contained in the closure of its interior.*

We will also define an involutive family $\mathbf{D}_0$, with which a family $\mathbf{F}_0$ of connected submanifolds will be associated. We use $\mathbf{A}_0(x, t)$ to denote the set of all $y \in \mathbf{R}^n$ that can be reached from x in exactly t units of time. We will prove:

Theorem 2. *The set* $\mathbf{A}_0(x, t)$ *is contained in* $\mathbf{F}_0(y)$, *where* y *is any element of* $\mathbf{A}_0(x, t)$. *Moreover, in the topology of* $\mathbf{F}_0(y)$, $\mathbf{A}_0(x, t)$ *is contained in the closure of its interior.*

Our results are a generalization of those of KUČERA [3], which correspond to the case $m = 1$, $A(u) = C + Bu$. As we explained in [5], a new proof of these results, based on different techniques, is of interest even for the case considered in [3]. The proof given here is based on the results of [6]. We remark that all our results, except for the remark at the end of Section 3, are valid if assumption (c) is weakened and $A(u)$ is taken to be an arbitrary real-analytic function of the u_i's (not necessarily a polynomial).

The organization of the paper is as follows: in Section 1 we give the basic definitions; in Section 2 we prove Theorems 1 and 2, and in Section 3 we apply these results to the study of the accessibility problem.

1. DEFINITIONS

We shall assume that the reader is familiar with the concepts of vector field, Lie bracket ($[X, Y] = XY - YX$) and submanifold (cf. HELGASON [2]). A vector field in $\mathbf{R}^n$ is viewed either as a derivation in the algebra of C^∞ functions in $\mathbf{R}^n$, or as a mapping from $\mathbf{R}^n$ into $\mathbf{R}^n$ (therefore, if X is a vector field in $\mathbf{R}^n$, and $x \in \mathbf{R}^n$, the notation $X(x)$ has an obvious meaning).

If $\mathbf{S}$ is any set of vector fields in $\mathbf{R}^n$, we shall write, for each $x \in \mathbf{R}^n$:

$$\mathbf{S}(x) = \{X(x) : X \in \mathbf{S}\} . \tag{4}$$

An *integral submanifold* of $\mathbf{S}$ is a connected submanifold S of $\mathbf{R}^n$ with the property that, for every $x \in S$, the tangent space S_x to S at x is the linear hull of $\mathbf{S}(x)$.

The set $\mathbf{S}$ is called involutive if

$$X \in \mathbf{S}\,, \quad Y \in \mathbf{S} \Rightarrow [X, Y] \in \mathbf{S}\,. \tag{5}$$

The following fundamental result is classical (Frobenius' theorem) if rank $\mathbf{S}(x)$ is constant (cf. [1]), and is proved without this assumption in [4].

Lemma 1. *Let* $\mathbf{S}$ *be an involutive family of analytic vector fields in* $\mathbf{R}^n$. *Then, through every point of* $\mathbf{R}^n$ *there passes a unique maximal (with respect to inclusion) integral submanifold of* $\mathbf{S}$.

With the control problem (1) we associate the following sets of matrices: $\mathbf{B}$ is the set of all matrices A_α; $\mathbf{B}'$ is the set of all "iterated Lie brackets" of elements of $\mathbf{B}$. Precisely, $\mathbf{B}'$ is the smallest set of matrices that contain all the brackets $[M, N]$ (M, N elements of $\mathbf{B}$) and that satisfies the condition

$$(6) \qquad M \in \mathbf{B}, \quad N \in \mathbf{B}' \Rightarrow [M, N] \in \mathbf{B}'$$

(the bracket $[M, N]$ of the matrices M and N is, by definition, the matrix $MN - NM$).

We let $\mathbf{B}_0$ denote the set of all matrices A_α, where α is any multiindex other than $(0, 0, \ldots, 0)$. Finally, we let

$$(7) \qquad \mathbf{E} = \text{linear hull of } \mathbf{B} \cup \mathbf{B}',$$

and

$$(8) \qquad \mathbf{E}_0 = \text{linear hull of } \mathbf{B}_0 \cup \mathbf{B}'.$$

Using the formulas $[M, N] = -[N, M]$ and $[M, [N, P]] = [[M, N], P] + [N, [M, P]]$ (Jacobi identity) it is easily shown that both $\mathbf{E}$ and $\mathbf{E}_0$ are Lie algebras of matrices (in fact, $\mathbf{E}_0$ is an ideal of $\mathbf{E}$).

With every $n \times n$ matrix M we can associate a vector field M^* on $\mathbf{R}^n$ defined by

$$(9) \qquad M^*(x) = Mx, \quad x \in \mathbf{R}^n.$$

The correspondence $M \to M^*$ is linear. Moreover

$$(10) \qquad [M^*, N^*] = [N, M]^*.$$

We define $\mathbf{D} = \{M^* : M \in \mathbf{E}\}$ and $\mathbf{D}_0 = \{M^* : M \in \mathbf{E}_0\}$. It is clear that $\mathbf{D}$ and $\mathbf{D}_0$ are involutive spaces of vector fields. We let $\mathbf{F}$ (resp. $\mathbf{F}_0$) be the family of all maximal integral manifolds of $\mathbf{D}$ (resp. $\mathbf{D}_0$). As in the introduction, the unique maximal integral submanifold of $\mathbf{D}$ (resp. $\mathbf{D}_0$) through a point $x \in \mathbf{R}^n$ is denoted by $\mathbf{F}(x)$ (resp. $\mathbf{F}_0(x)$).

We now define the main controllability concepts. A *control* is an m-tuple $u = (u_1, \ldots, u_m)$ of piecewise continuous functions defined on an interval $[0, t_u]$, and such that $0 \leqq u_i(t) \leqq 1$ for $0 \leqq t \leqq t_u$, $i = 1, \ldots, m$. If u is a control, and if $x_0 \in \mathbf{R}^n$, there is a unique solution $t \to x(t)$ of the system

$$(11) \qquad \dot{x}(t) = A(u(t))\, x(t)$$

which satisfies the initial condition $x(0) = x_0$, and is defined for $0 \leqq t \leqq t_u$. The value at t of this solution is denoted by $\pi(x_0, u; t)$.

If x and y are vectors in $\mathbf{R}^n$, we say that y is *attainable* from x in t units of time if there exists a control u such that $\pi(x, u, t) = y$. The set of all y that are attainable from x in t units of time will be denoted by $\mathbf{A}_0(x, t)$. The union of the sets $\mathbf{A}_0(x, s)$ for $0 \leqq s \leqq t$ is denoted by $\mathbf{A}(x, t)$.

2. PROOF OF THE MAIN RESULTS

We shall apply to the system (1) the results of [6]. We must consider the family D of all vector fields of the form $A(u)^*$, where u belongs to the cube

$$C = \{(u_1, \ldots, u_m) : 0 \leqq u_i \leqq 1 \text{ for } i = 1, \ldots, m\} .$$

We want to apply Theorem 4.4 of [6]. We must therefore determine $\mathscr{I}(D)$.

Lemma 2. $\mathscr{I}(D) = \mathbf{D}$.

Proof. In view of the definition of $\mathbf{D}$, and of the fact that $\mathscr{I}(D)$ is the smallest involutive subspace that contains D, our results will follow if we prove: the linear hull of the matrices $A(u)$, $u \in C$ coincides with the linear hull of $\mathbf{B}$. But this is immediate: every $A(u)$ is a linear combination of the A_α's; conversely, every A_α can be obtained as a derivative (of a sufficiently high order) of the function $u \to A(u)$, at $u = (0, \ldots, 0)$. Therefore, A_α belongs to the linear hull of the $A(u)$, $u \in C$, and our lemma is proved.

Theorem 1 now follows from Theorem 4.4 of [6].

In order to prove Theorem 2, we want to apply Theorem 4.5 of [6]. This requires that we compute $\mathscr{I}_0(D)$.

Lemma 3. $\mathscr{I}_0(D) = \mathbf{D}_0$.

Proof. Recall that $\mathscr{I}_0(D)$ is the set of all vector fields $X + Y$, where $Y \in \mathscr{I}'(D)$ (the derived algebra of $\mathscr{I}(D)$), and where $X \in D_0$ (the set of all linear combinations $\Sigma\lambda_i X_i$ such that the X_i's belong to D and that $\Sigma\lambda_i = 0$). Now, it is easy to see that $\mathscr{I}'(D)$ is precisely the linear hull of the vector fields M^*, $M \in \mathbf{B}'$. It follows that it is sufficient to prove that the linear hull of the vector fields M^*, $M \in \mathbf{B}_0$, coincides with D_0, But D_0 is the linear hull of the differences $X - Y$, $X \in D$, $Y \in D$. Therefore, it is sufficient to prove that the linear hull L_1 of the matrices $A(u) - A(v)$, $u \in C$, $v \in C$, coincides with L_2, the linear hull of $\mathbf{B}_0$. First, every $A(u) - A(v)$ is a linear combination of the A_α with $\alpha \neq (0, \ldots, 0)$. This shows that L_1 is contained in L_2. To prove the converse let, for each $u \in C$, $A'(u) = A(u) - A_{(0\ldots,0)}$. Then $A'(u) = A(u) - A(v)$, where $v = (0, \ldots, 0)$. It follows that $A'(u) \in L_1$ for every $u \in C$. By repeated partial differentiation, we conclude that $A_\alpha \in L_1$ for each $\alpha \neq (0, \ldots, 0)$. Thus $L_2 \subset L_1$, and the proof is complete.

Theorem 2 is now an immediate consequence of Theorem 4.5 of [6].

3. ACCESSIBILITY

Recall that a control system is said to have the accessibility property from x if the set $\mathbf{A}(x, t)$ has a nonempty interior for some $t > 0$ (cf. [6]). In this case, the interior of $\mathbf{A}(x, t)$ will be nonempty for every $t > 0$ (provided an analyticity condition is satisfied).

Our results yield the following criterion for accessibility:

Corollary 1. *The system* (1) *has the accessibility property from* x *if and only if the set of all vectors* Mx, $M \in \mathbf{B} \cup \mathbf{B}'$, *has rank* n.

The strong accessibility property is defined in a similar way, with $\mathbf{A}(x, t)$ replaced by $\mathbf{A}_0(x, t)$ (cf. [6]). We have:

Corollary 2. *The system* (1) *has the strong accessibility property from* x *if and only if the set of all vectors* Mx, $M \in \mathbf{B}_0 \cup \mathbf{B}'$, *has rank* n.

We remark that our previous results imply that the accessibility and strong accessibility properties depend only on the set of coefficients of $A(u)$. As an illustration, assume that the system $\dot{x} = (u_1 A_1 + u_2 A_2 + u_3 A_3)\, x$ $(0 \leqq u_i \leqq 1,\ i = 1, 2, 3)$, in which there are three controls that can be varied independently, has the accessibility property from a certain point $x_0 \in \mathbf{R}^n$. It follows immediately from Cor. 1 that the system $\dot{x} = (A_1 + uA_2 + u^2 A_3)\, x$ $(0 \leqq u \leqq 1)$, in which only one control is available, will also have the accessibility property from x_0.

Finally, we observe that the criteria of Cors. 1 and 2 are "effective" in the following sense: given the matrices A_α, one can determine in a finite number of steps whether the condition of Cor. 1 or Cor. 2 holds. For instance, one can check whether or not the condition of Cor. 1 holds as follows: let $n(k)$ be the rank of the set vectors Mx, where M ranges over all the brackets of k or less elements of $\mathbf{B}$. One can successively compute $n(0), n(1), \ldots$ (each computation requires finitely many steps). Eventually a k will be reached such that $n(k) = n(k + 1)$. It is easy to show that this implies $n(k + 1) = n(k + 2) = \ldots$, and, therefore, the condition of Cor. 1 holds if and only if $n(k) = n$.

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