Motupalli Satyanarayana
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Czechoslovak Mathematical Journal, Vol. 22 (1972), No. 4, 509–516

CZECHOSLOVAK MATHEMATICAL JOURNAL
Mathematical Institute of Czechoslovak Academy of Sciences
V. *22* (*97*), PRAHA 16. 11. 1972, No 4

COMMUTATIVE PRIMARY SEMIGROUPS

M. SATYANARAYANA, Bowling Green

(Received March 18, 1970)

A commutative semigroup is said to be primary if every ideal in the semigroup is primary. In this paper we shall study the structure of primary semigroups. We observe that the structure of primary semigroups differs from the structure of commutative primary rings, which are studied by the author in [2]. In the ring of integers, non-zero prime ideals are maximal and the cancellative property in the semigroup – theoretic sense is satisfied. But every ideal is not primary. In contrast to this, it is proved in 2.11 that every cancellative semigroup containing identity is primary iff every non-zero prime ideal is maximal. We also characterize cancellative primary semigroups without identity. It is shown that in every primary semigroup idempotents form a chain under the usual natural ordering and also prime ideals form a chain under set-inclusion. It is natural to ask whether a commutative semigroup is primary if the idempotents form a chain. This need not be true [Example 2.3]. In the case of commutative regular semigroup, it is shown in 2.4, that the primary condition is equivalent to the property that idempotents are linearly ordered. Most of the results of this paper are obtained by using the concept of the radical. This theory of radical in semigroup is nothing but the new version of the existing theory of radical in commutative ring theory. This theory has been sketched in section 1 without proofs.

1. IDEAL THEORY

Throughout this paper all semigroups under consideration are commutative. An ideal A is a semigroup S is primary whenever $xy \in A$, $x, y \in S$ and $x \notin A$, then there exists a positive integer n such that $y^n \in A$. An ideal A in S is called a prime ideal whenever $xy \in A$ and $x \notin A$, then $y \in A$. The semigroup S is itself a prime ideal in S. An ideal A is said to be maximal iff it is not contained in a proper ideal other than S itself. Radical of an ideal A in a semigroup S, denoted by $\sqrt{A}$, is the set of all $x \in S$ such that some power of x is in A. $\sqrt{A}$ is an ideal and contains A. The propositions and proofs here under broadly follow the same lines as for commutative rings. For this reason we do not mention any proposition with a formal label such as Theorem etc., but however merely number them for the sake of cross-reference. In order to avoid any possible confusion, we shall mention here the proofs in case there is a variation, even by little, from the proofs of the analogue theorems in commutative rings found in [1] and [3]. "$\subseteq$" denotes containment and "$\subset$" proper containment.

1.1. *If A and B are any two ideals in a commutative semigroup S, then*

i) $A \subseteq B \Rightarrow \sqrt{A} \subseteq \sqrt{B}$,

ii) $\sqrt{(AB)} = \sqrt{(A \cap B)} = \sqrt{A} \cap \sqrt{B}$,

iii) $\sqrt{(\sqrt{A})} = \sqrt{A}$.

1.2. *If A is a primary ideal, then $\sqrt{A}$ is a prime ideal. Moreover if $ab \in A$ and $a \notin A$, then $b \in \sqrt{A}$. If P is a prime ideal, $\sqrt{P} = P$. More generally an ideal A and its radical are contained in precisely the same prime ideals.*

1.3. *Let A be an ideal in a semigroup with identity. If $\sqrt{A} = M$, where M is a maximal ideal, then A is a primary ideal.*

Proof. S has a unique maximal ideal, which is the union of all proper ideals. Hence M is the unique maximal ideal and every element of S not belonging to M is an unit. Let $xy \in A$ and $x \notin A$. If no power of y is in A i.e., $y \notin M$, then y has an inverse y^{-1}. Hence $x = xy\, y^{-1} \in A$, a contradiction.

1.4. *In a commutative semigroup with identity, the unique maximal ideal M is prime; $\sqrt{M^n} = M$ for every positive integer n and M^n is a primary ideal for every positive integer n.*

1.5. *In a commutative semigroup without identity, maximal ideals need not be prime. Consider a semigroup $S = \{a, b, z\}$ with the following multiplication table.*

	z	a	b
z	z	z	z
a	z	a	a
b	z	a	a

$\{z, a\}$ is a maximal ideal but not prime since $b \,.\, b \in \{z, a\}$ but $b \notin \{z, a\}$.

1.6. Definition. Let A be an ideal in a commutative semigroup S. Then a prime ideal P is said to be *a minimal prime ideal belonging to A* if $A \subseteq P$ and there is no prime ideal Q in S such that $A \subseteq Q \subset P$.

1.7. Definition. A subset T in a semigroup S is called *a multiplicative system* if, for $x, y \in T$, $xy \in T$. T is nothing but a subsemigroup.

1.8. If two subsets of a semigroup have no elements in common, we say that either of these sets does not meet other.

1.9. [1; 104]. *Let A be an ideal in commutative semigroup S and M is a subsemigroup of S which does not meet A. Then M is contained in a maximal subsemigroup M^* which does not meet A, that is, if N is a subsemigroup such that $M^* \subset N$, then N contains an element of A.*

1.10. [1; 105]. *Let M be a subsemigroup in a commutative semigroup S and A an ideal which does not meet M. Then A is contained in a maximal ideal P which does not meet M, that is, if N is an ideal such that $P \subset N$, then N contains an element of M. Such an ideal P is necessarily prime.*

Proof. Let J be the collection of ideals B such that $B \supseteq A$ and B does not meet M J is non-empty and inductive. Then by Zorn's lemma there exists a maximal element P of J. We claim that P is prime. Let $xy \in P$ and x and $y \notin P$. Since $P \cup (x \cup xS)$ and $P \cup (y \cup yS)$ contain P properly there exist a and b such that $a \in [P \cup (x \cup xS)] \cap$ $\cap M$ and $b \in [P \cup y \cup ys] \cap M$. Since P does not meet M by the choice of P, we have $a \in (x \cup xS) \cap M$ and $b \in (y \cup yS) \cap M$. Then $ab \in xy \cup xyS \subset P$. But $ab \in M$, since M is a subsemigroup. So $ab \in P \cap M \neq \emptyset$, a contradiction.

1.11. Notation. If A is a subset of a semigroup S, then $S - A$ denotes the complement of A in S.

1.12 [1; 106]. *A set P of elements of the commutative semigroup S is a minimal prime ideal belonging to the ideal A iff $S - P$ is a maximal subsemigroup which does not meet A.*

1.13 [1; 104]. *The radical of an ideal A in the commutative semigroup is the intersection of all minimal prime ideals belonging to A or equivalently is the intersection of all prime ideals containing A.*

1.14 [1; 107]. *Any prime ideal containing an ideal A in a semigroup contains a minimal prime ideal belonging to A.*

2. PRIMARY SEMIGROUPS

2.1. Proposition. *In a commutative primary semigroup S,* i) *prime ideals form a chain under set-inclusion* ii) *idempotents form a chain under natural ordering.*

Proof. If A and B are any two prime ideals, then $\sqrt{(A \cap B)}$ is prime since the radical of a primary ideal is prime by 1.2. Then $A \cap B = \sqrt{A} \cap \sqrt{B} = \sqrt{(A \cap B)}$ is prime. If $A \not\subset B$ or $B \not\subset A$, then there exist $a \in A$ and $\notin B$ and $b \notin A$, $b \in B$. Then $ab \in A \cap B$ but $a, b \notin A \cap B$. Hence $A \subseteq B$ or $B \subseteq A$. To prove (ii): Let e and f be any two idempotents in S. Since $\sqrt{eS}$ and $\sqrt{fS}$ are prime ideals by 1.2, we have by (i), $\sqrt{eS} \subseteq \sqrt{fS}$ or $\sqrt{fS} \subseteq \sqrt{eS}$. Thus $e \in fS$ or $f \in eS$ i.e., $e = fe$ or $f = ef$. This implies under the usual natural ordering, the idempotents form a chain.

2.2. Note. If a semigroup has 0 and if we assume that every non-zero ideal is primary, condition (i) in 2.1 need not be true. This can be observed in the semigroup $S = \{0, a, b, 1\}$ where $ba = ab = 0$, $a^2 = a$, $b^2 = b$ with 0 and 1 as zero and identity of S respectively. The prime ideals $\{0, a\}$ and $\{0, b\}$ are not contained in one another.

2.3. Note. If the idempotents are linearly ordered, then the semigroup need not be primary. Consider the semigroup $S = \{0, 1, a, b\}$ where $a^2 = ab = ba = 0$, $b^2 = b$ with 0 and 1 as the zero and identity of S respectively. Since (0) is not primary, S is not a primary semigroup. Trivially the idempotents form a chain. Now we shall show that for certain class of semigroups, the condition that the semigroup is primary is equivalent to the statement that idempotents are linearly ordered. The example in 2.2 asserts that even in bands idempotents need not form a chain. In the following theorem, my colleague, Dr. McMorris showed me (iv) $\Rightarrow$ (ii). However we shall mention a different proof, originally developed by me for idempotent semigroups.

2.4. Theorem. *Let S be a regular commutative semigroup. Then the following statements are equivalent.*

i) *Every ideal in S is prime.*
ii) *S is a primary semigroup.*
iii) *The idempotents form a chain.*
iv) *The ideals are linearly ordered.*

Proof: Trivially (i) $\Rightarrow$ (ii). By 2.1, (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (iv): It suffices to show that principal ideals are linearly ordered. For, if A and B are any two ideals and if $A \nsubseteq B$, then there exists $a \in A$ and $\notin B$. If principal ideals are linearly ordered, then for any $b \in B$, we have $b \cup bS \subseteq a \cup aS$. Hence $B \subseteq A$. Consider now any two principal ideals in S. They should be of the form eS and fS where e and f are idempotents. Since $e = ef$ or $f = ef$ by hypothesis, we have either $eS \subseteq fS \subseteq eS$.

(iv) $\Rightarrow$ (i): Let A be an arbitrary ideal in S. Suppose $xy \in A$. Since S is regular, $x \cup xS = eS$ and $y \cup yS = fS$ where e and f are idempotents. Clearly $xy \cup xyS =$

$= efS$. Then $ef \in A$. Since ideals are linearly ordered, assume $eS \subseteq fS$, which implies $e = ef \in A$ i.e., $x \in A$. Hence x or $y \in A$. Thus A is a prime ideal.

Now in the following theorem we cite a wide class of primary semigroups.

2.5. Theorem. *Let S be a commutative semigroup with identity. If (non-zero: assume this if S has 0) prime ideals are maximal, then S is a primary semigroup.*

Proof. Recall 1.13 that radical of an ideal is the intersection of all prime ideals that contain the ideal. If S is not a group, S has ideals and so S has a unique maximal ideal M, which is the union of all proper ideals in S. By hypothesis M is the only (non-zero) prime ideal. If A is a (non-zero) ideal, then $\sqrt{A} = M$ and hence A is primary by 1.3. If S has zero and if (0) is a prime ideal, (0) is primary and hence S is primary. If (0) is not a prime ideal, $\sqrt{(0)}$ is the set of all nilpotents and hence is M itself by 1.13. Thus S is a primary semigroup.

2.6. Remark. The converse of 2.5 need not be true. Consider $S = \{1, a, b\}$ where $a^2 = ab = ba = a$, $b^2 = b$ and $1 . s = s . 1 = s$ for every $s \in S$. S is a primary semigroup in which the prime ideal (a) is not maximal. The theorem 2.5 is not true in the case of commutative rings. In the ring of integers non-zero prime ideals are maximal but every ideal is not primary. For example the ideal (12) is not primary since $3.4 \in (12)$ but $3 \notin (12)$ and no power of 4 is in (12). However in the case of cancellative semigroups, the converse of 2.5 can be proved to be true. For this we require the following results.

2.7. Theorem. *Let S be a cancellative commutative semigroup. If S is either a primary semigroup or a semigroop in which an ideal is primary iff its radical is prime, then a primary ideal Q in S is prime and $Q = \sqrt{Q}$ in case $\sqrt{Q}$ is a non-maximal prime ideal.*

Proof. Since $\sqrt{Q}$ is a non-maximal ideal, we have $\sqrt{Q} \subset A \subset S$, where A is an ideal in S. Let $a \in A - \sqrt{Q}$ and $b \in \sqrt{Q}$. Now $Q \subset Q \cup ab \cup abS \subset \sqrt{Q}$. This implies $\sqrt{Q} \subset \sqrt{(Q \cup ab \cup abS)} \subset \sqrt{Q}$ by 1.1. Hence $\sqrt{(Q \cup ab \cup abS)} = \sqrt{Q}$. Let $s \in S - A$. Then $sab \in Q \cup ab \cup abS$, since $a \notin \sqrt{Q} = \sqrt{(Q \cup ab \cup abS)}$ and $Q \cup ab \cup abS$ is primary by virtue of the hypothesis, $bs \in Q \cup ab \cup abS$ by 1.2. If $bs \in ab \cup abS$, then $bS = abt$ or ab and hence by cancellative property we have $s = a$ or $s = at$, i.e., $s \in A$, a contradiction. Thus $bs \in Q$, which implies $b \in Q$ since $s \notin \sqrt{Q}$ by 1.2. Hence $\sqrt{Q} \subseteq Q$ and so $Q = \sqrt{Q}$. Thus Q is prime.

2.8. Proposition. *Let S be cancellative commutative semigroup. Suppose S is a primary semigroup or a semigroup in which an ideal is primary iff its radical is prime. If P is a minimal prime ideal containing $d \cup dS$, $d \in S$, then P is a maximal ideal.*

Proof. Let P be not a maximal ideal. Assume $M = S - P$ and $A = \{x \in S \mid xm \in \in d \cup dS$ for some $m \in M\}$. Clearly A is an ideal. Now $x \in A \Rightarrow xm \in P \Rightarrow x \in P$ since $m \notin P$. Thus $A \subseteq P$ we claim $P = A$. Let $b \in P$ and suppose $N = \{b^k m, k \geqq 0, m \in M\}$. N is a subsemigroup and N contains the subsemigroup M properly since $bm \in N$ and $bm \notin M$. Since P is a minimal prime ideal of $d \cup dS$, M is maximal subsemigroup not meeting $d \cup dS$ by 1.12. Since N contains M properly $N \cap d \cup dS$ is not empty. So there exists $k > 0$ such that $b^k m \in d \cup dS$. This implies $b^k \in A$ and thus $b \in \sqrt{A}$. Therefore $P \subseteq \sqrt{A} \subseteq \sqrt{P} = P$. So $P = \sqrt{A}$. But by hypothesis A is primary. Since P is not maximal, we have by 2.7, $P = A$. P is also a minimal prime ideal of $d^2 \cup d^2S$. Let $B = \{y \in S \mid ym \in d^2 \cup d^2S$ for some $m \in M\}$. As before we have $B = P$. Since $d \in P = A = B$, we have $dm = sd^2$ for some $s \in S$ or $dm = d^2$. Then, by cancellative property, $m = sd$ or $d \in d \cup dS \subseteq P$, a contradiction.

2.9. Theorem. *If S is a cancellative commutative semigroup such that either S is primary or in S an ideal A is primary iff $\sqrt{A}$ is a prime ideal, then the prime ideals are maximal.*

Proof. Let P be a prime ideal in S. Suppose $d \in P$. Then by 1.14, $d \cup dS$ is contained in a minimal prime ideal, which is maximal by 2.8. Hence P is maximal ideal.

2.10. Remark. If cancellation condition is dropped, 2.9 need not be true. In the example 1.5 every ideal is primary. But $\{z\}$ is prime but not maximal.

2.11. Theorem. *Let S be a cancellative commutative semigroup with identity. Then the following statements are equivalent.*

i) *Prime ideals are maximal.*

ii) *S is a primary semigroup.*

iii) *If A is an ideal, A is primary iff $\sqrt{A}$ is prime.*

iv) *If x and y are non-units, then there exist positive integers n and m such that $x^n = ky$, $k \in S$ and $y^m \in lx$, $l \in S$.*

Proof. Combining 2.9 and 2.5 we have (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).

(ii) $\Rightarrow$ (iv): Since S has identity, S has a unique maximal ideal M, which is the only prime ideal in S. By 1.13 radical of an ideal is the intersection of all prime ideals that contain the ideal. Hence if x and y are any two non-units in S, then $\sqrt{xS} = \sqrt{yS} = M$, since x and $y \in M$. Hence (iv) follows. It is trivially true that (iv) $\Rightarrow$ (ii).

2.12. Remark. If the semigroup in 2.11 has zero, then 2.11 is true by assuming non-zero prime ideals are maximal.

2.13. Theorem. *Let S be a cancellative commutative semigroup not containing the identity. Then the following are equivalent.*

i) *S is a primary semigroup.*

ii) *S has no proper prime ideals.*

iii) *If $x, y \in S$, then there exist positive integers n and m such that $x^n = ky$ and $y^m = lx$, where $k, l \in S$.*

Proof. (ii) $\Rightarrow$ (iii): For any $x, y \in S$, we have $\sqrt{(x \cup xS)} = \sqrt{(y \cup yS)} = S$. Hence (iii) is evident. Trivially (iii) $\Rightarrow$ (i). To prove (i) $\Rightarrow$ (ii): Since prime ideals are maximal by 2.9 and since prime ideals form a chain under set inclusion by 2.1, S has either no proper prime ideals or a unique proper prime ideal, which is maximal. We claim that the latter case is not possible. Assume that M is the unique prime ideal, which is also maximal. If $a \notin M$, then $a^2 \notin M$ since M is prime. So $S = M \cup a \cup aS = M \cup a^2 \cup a^2S$. Then for any $a \in S - M$, $a = a^2$ or $a = a^2x$ for some $x \in S$. Clearly a and ax are in $S - M$. In either case $S - M$ has idempotents. Since S is cancellative S has at most one idempotent, say e. Then $S - M$ is a group. Also, since e is not in any proper prime ideal, $\sqrt{(e \cup eS)} = S$. This implies for $x \in M$, we have $x^n = e$ or $x^n = ex^n$. The first is not possible since $e \notin M$. The second implies $x = ex$ by cancellative property. Thus e is the identity of S, a contradiction.

2.14. Theorem. *Let S be a cancellative commutative semigroup. Then S is primary iff prime ideals in S are maximal. Furthermore S has no idempotents except identity, if it exists.*

Proof. Clearly by 2.9 prime ideals are maximal in primary semigroups. Assume now that prime ideals are maximal. If S has identity, by 2.5 S is a primary semigroup. Let S have no identity. If S has no proper prime ideals, an easy verification yields that S is primary. We claim that the case that S has proper prime ideals does not arise. Suppose S has proper prime ideals. Then S has a unique prime ideal. For, let S have prime ideals P and Q. As in the proof of 2.13, $S - P$ and $S - Q$ are groups. Then the identity e of $S - P$ and $S - Q$ should be the same since a cancellative semigroup can have at most one idempotent. Now $x \in S - P \Rightarrow \exists y$ such that $xy = e \in S - Q$. If $x \notin S - Q$, $xy = e \in Q$, a contradiction. Thus $x \in S - Q$. Hence by symmetry $S - P = S - Q$ and thus $P = Q$. If e is the identity of $S - P$, then eS is a prime ideal and hence $eS = S$. This implies that e is the identity of S, acont radiction.

To prove the last part observe that S has at most one idempotent since S is cancellative. Suppose that S has no identity. If e is an idempotent, then eS is a prime ideal. But we have proved above that S has no proper prime ideals. So $eS = S$ and thus e is the identity.

2.15. Remark. The above theorem need not be true for non-cancellative semigroups. Let $S = \{a, b, ab : a^2 = a;\ b^2 = b;\ ab = ba\}$. The only prime ideals in S are $\{a, ab\}$ and $\{b, ab\}$ which are maximal. But the ideal $\{ab\}$ is not primary.

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Author's address: Bowling Green State University Ohio 43402, U.S.A.