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ON CANONICAL FORMS ON NON-HOLONOMIC AND SEMI-HOLONOMIC PROLONGATIONS OF PRINCIPAL FIBRE BUNDLES

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Using the theory of jets, Kolář [3] introduced canonical forms on holonomic prolongations of principal fibre bundles and justified the algorithm for the determination of their structure equations. (This algorithm was also used by Laptev [6] and Gheorghiev [2].) In the present paper, using Kolář's results, we shall find the structure equations of canonical forms on non-holonomic and semi-holonomic prolongations of principal fibre bundles as well as the Maurer-Cartan equations of the structure groups of these bundles. In particular, we shall show that the structure equations of "the non-holonomic differential group of order r" introduced by Lumiste in [9] (see also [7]) are the structure equations of the group $L_1^r$ of all invertible semi-holonomic r-jets of $R^n$ into $R^n$ with the source and the target $O$. Further, the semi-holonomic extensions of the canonical form $\Theta_1$ of $W^1(P)$ are introduced and some relations to the theory of linear connections are explained. We shall use the terminology and notation of the theory of jets (see [1]) throughout the paper. Our considerations are in the category $C^\infty$.

1. Let $M_1, M_2$ be manifolds, $n_i = \dim M_i$, $i = 1, 2$. As usual, $J^r_x(M_1, M_2)$ or $J^r_x(M_1, M_2)$ denotes respectively the manifold of all non-holonomic or semi-holonomic r-jets of $M_1$ into $M_2$ with source $x \in M_1$. Let $U$ be a coordinate neighbourhood on $J^r_x(M_1, M_2)$ and let

$$c^j_{i_1, \ldots, i_r}, \quad j = 1, 2, \ldots, n_2; \quad i_1, \ldots, i_r = 0, 1, \ldots, n_1$$

be coordinate functions on $U$, see [12]. Denote by $\zeta$ the rule of dropping all the zero components in a multiindex. Vîrsîk [12] deduced the following property of semi-holonomic jets.
Lemma 1. Let \( \tilde{f} \in U \subseteq J^*_r(M_1, M_2) \). Then \( \tilde{f} \) is semi-holonomic if and only if its coordinates \( c^i_{i_1 \ldots i_r} (\tilde{f}) \) satisfy
\[
\zeta(i_1 \ldots i_r) = \zeta(k_1 \ldots k_r) \Rightarrow c^i_{i_1 \ldots i_r} (\tilde{f}) = c^i_{k_1 \ldots k_r} (\tilde{f}) .
\]

2. In what follows we shall use the following indices: \( A, B = 1, \ldots, n + m \); \( \alpha, \beta, \gamma = 1, \ldots, n + m \); \( h, t, p = 1, \ldots, n \); \( i, j, k = 0, 1, \ldots, n \).

Let \( P(B, G, \pi) \) be a principal fibre bundle, \( n = \text{dim } B \), \( m = \text{dim } G \). Let \( U \subseteq R^n \) be an open subset, \( 0 \in U \) and let \( V \) be an open subset of \( B \). A local isomorphism \( \Psi^{\phi_\sigma} : U \times G \to \pi^{-1}(V) \), \( \Psi^{\phi_\sigma}(x, g) = [\sigma(\phi(x))] g \), will be called an allowable chart on \( P \), where \( \phi \) is a diffeomorphism \( U \to V \) and \( \sigma \) is a local cross-section \( V \to \pi^{-1}(V) \).

The first prolongation of \( P \) is the set \( W^1(P) \) of all 1-jets of allowable charts on \( P \) with the source \( (0, e) \in R^n \times G \). \( W^1(P) \) is a principal fibre bundle over \( B \) with the structure group \( G_n^1 \) of all 1-jets of allowable charts \( \Psi^{\phi_\sigma} \) on \( R^n \times G \) with the source \( (0, e) \) satisfying \( \phi(0) = 0 \). We can identify \( W^1(P) \equiv H^1(B) \otimes J^1(P) \) where \( H^1(B) \otimes J^1(P) \) means the fibre product over \( B \) of \( H^1(B) \) and \( J^1(P) \) and \( G_n^1 \) is the semidirect product \( L^1_n \times T_n^1(G) \) of the groups \( L^1_n \) and \( T_n^1(G) \) with respect to the action \( S \mapsto SX(S \in T_n^1(G), X \in L^1_n) \) of \( L^1_n \) on \( T_n^1(G) \). In other words,
\[
(1) \quad (X_2, S_2)(X_1, S_1) = (X_2X_1, (S_2X_1)S_1)
\]
where \( X_2X_1, S_2X_1 \) is the composition of jets and \( (S_2X_1)S_1 \) is the product in the group \( T_n^1(G) \). By induction, we define \( \hat{W}^r(P) = W^1(\hat{W}^{r-1}(P)) \), and call it the \( r \)-th non-holonomic prolongation of \( P \). \( \hat{W}^r(P) \) has a natural structure of a principal fibre bundle \( \hat{W}^r(P)(B, \hat{G}_n^r) \), where the group \( \hat{G}_n^r \) is determined by the recurrent formula \( \hat{G}_n^r = (\hat{G}_n^{r-1})^1 \).

We can identify
\[
\hat{W}^r(P) = \hat{H}^r_n(B) \otimes J^r(P), \quad \hat{G}_n^r = L^r_n \times T_n^r(G), \quad \text{see [11] and [4]}.
\]

This identification will be denoted by \( \tau \). It results from the definition of \( \hat{W}^r(P) \) that \( \hat{W}^r(P) \) is the set of all 1-jets of the local isomorphisms \( \Psi^{\phi_\sigma} \) of \( R_n \times \hat{G}_n^{r-1} \) into \( \hat{W}^{r-1}(P) \) with the source \( (0, e_{\tau - 1}) \), where \( e_{\tau - 1} \) is the unit of the group \( \hat{G}_n^{r-1} \). These local isomorphisms will be called non-holonomic allowable \( (r - 1) \)-charts on \( P \).

Let \( u \in \hat{W}^r(P) \), \( u = j^1_{(0, e_{\tau - 1})}^{u} \Psi \), where \( ^u \Psi \) is a non-holonomic allowable \( (r - 1) \)-chart on \( P \). Denote by \( \beta \) the natural projection \( \beta u = ^u \Psi(0, e_{\tau - 1}) \). Let \( X \in T_n(\hat{W}^r(P)) \), \( X = j^1_0 \gamma(t) \). Then \( ^u \Psi^{-1}_{\tau} \beta \ast X = j^1_0 \Psi^{-1}_{\tau}[\beta \gamma(t)] \in T_{(0, e_{\tau - 1})}(R^n \times \hat{G}_n^{r-1}) \equiv \hat{w}^{r-1}(G) \). Thus we get a vector-valued form on \( \hat{W}^r(P) \) with values in \( \hat{w}^{r-1}(G) \)
\[
(2) \quad \Theta_r(X) = ^u \Psi^{-1}_{\tau} \beta \ast (X).
\]

Definition 1. The form \( \Theta_r \) determined by (2) will be said to be the canonical form on \( \hat{W}^r(P) \).
In [4], Kolář introduced an admissible extension of $\Theta_r$ in the holonomic case. Put $\hat{W}^{r+1}(P) = W^2(W^{r-1}(P))$, i.e., $\hat{W}^{r+1}(P)$ is the second holonomic prolongation of the principal fibre bundle $W^{r-1}(P)$. According to Lemma 3 of [4], $\hat{W}^{r+1}(P)$ is a principal fibre bundle $\hat{W}^{r+1}(P)/(W^r(P), \sigma(u) = j^2_{(0,\sigma^{-1})})$, where $\sigma(W^{r-1})$ is the kernel of the homomorphism $j^2_{(0,\sigma^{-1})}$ is the natural projection of 2-jets into 1-jets. Since $\sigma(W^{r-1})$ is homeomorphic to a number space, the global sections of $\hat{W}^{r+1}(P)$ exist. Let $\sigma$ be a global section of $\hat{W}^{r+1}(P)$; $u \in \hat{W}^{r}(P), \sigma(u) = j^2_{(0,\sigma^{-1})}$, where $\Psi$ is a non-holonomic allowable $(r - 1)$-chart; $X \in T_u(\hat{W}^{r}(P)), X = j^1_{0}(\gamma(t))$. Denote by $\Psi^{-1}(\gamma(t))$ the image of the jet $\gamma(t)$ by the map $\Psi^{-1}$. Then $j^1_{0}(\gamma(t)) \in \hat{w}^{r}_a(G)$. The form

\[ \Theta(X) = j^1_{0}(\gamma(t)) \]

will be said to be an admissible extension of the canonical form $\Theta_r$ on $\hat{W}^{r}(P)$. One can see easily that

\[ (j^r_{-1})^* \Theta_r(X) = \Theta_r(X). \]

3. Let $e^a$ be a basis of $g$ (as usual $g$ denotes the Lie algebra of $G$). Let $\omega^e$ be the dual basis of $g^*$ and let

\[ d\omega^e = \frac{1}{2}c^e_{\rho\gamma} \omega^\rho \wedge \omega^\gamma \]

be the structure equations of $G$. Let $e^a$ be the canonical coordinates in some neighbourhood of $e \in G$ determined by $e^a$, and let $e^A$ be canonical coordinates on $R^n$. Denote by $e^A_{i_1...i_r}$ the corresponding local coordinates on $R^n \times G$. Let $e^A_{i_1...i_r}$ be the corresponding basis of $\hat{w}^r_a(G)$. The space $\hat{w}^{r-1}_a(G)$ is isomorphic to the subspace $\{e^A_{i_1...i_{r-1}}\} \subset \hat{w}^{r}_a(G)$. Taking into account this isomorphism and (3), we can write

\[ \Theta(X) = \Theta^A_{i_1...i_r-1} \epsilon^{i_1...i_{r-1}}, \quad \Theta'(X) = \Theta(X) + \Theta^A_{i_1...i_{r-1}}(X) e^A_{i_1...i_{r-1}}. \]

Kolář, [3], deduced the following structure equations of $\Theta_1$:

\[ d\Theta_0^p = \Theta_0^q \wedge \Theta_0^p, \quad d\Theta_0 = \frac{1}{2}c^A_{\rho\gamma} \Theta_0^\rho \wedge \Theta_0^\gamma + \Theta_0^\rho \wedge \Theta_0^\gamma. \]

The relations (4) do not depend on the choice of an admissible extension of $\Theta_1$ (see [4], Theorem 5). The exterior differentiation of (4) yields

\[ d\Theta^A_0 = \Theta^B_q \wedge \Theta^A_q + \Theta^A_0 \wedge \Theta^A_0, \]

where the forms $\Theta^A_0$ satisfy $\Theta^A_0 \wedge \Theta^A_0 \wedge \Theta^A_0 = 0, \Theta^A_0 = 0, \Theta^A_0 = c^A_{\rho\gamma} \Theta_0^\rho \wedge \Theta_0^\gamma$. This implies the following structure equations of $G^1$ (see [4]):

\[ d\omega^A_0 = \frac{1}{2}c^A_{\rho\gamma} \omega_0^\rho \wedge \omega^A_0, \quad d\omega^A_0 = \omega_0^B \wedge \omega^A_0, \]

where $\omega_0^A = c^A_{\rho\gamma} \omega_0^\rho, \omega_0^A = 0$ and $\omega^A_0, \omega^A_0$ form the basis of $\hat{w}^1_a$ dual to $e^0, e^A$.
Proposition 1. (Structure equations of the canonical form \( \Theta_r \)) Let \( \Theta_{i_1 \ldots i_r} \) be the components of \( \Theta'_r \). Then

\[
\begin{align*}
\text{d} \Theta_{0 \ldots 0}^z &= \Theta_{0 \ldots 0}^z \wedge \Theta_{0 \ldots 0}^z_0 \\
\text{d} \Theta_{0 \ldots 0}^p &= \frac{1}{2} c_{p}^z \Theta_{0 \ldots 0}^z \wedge \Theta_{0 \ldots 0}^p + \Theta_{0 \ldots 0}^p \wedge \Theta_{0 \ldots 0}^{z_0} 
\end{align*}
\]

Further, if \( i_p \) is the first number different from 0 in the sequence \( (i_1, i_2, \ldots, i_r) \), then

\[
\begin{align*}
\text{d} \Theta_{i_1 \ldots i_p \ldots i_r} &= \sum_{(k_p + 1 \ldots k_r)} \Theta_{0 \ldots 0}^{i_p} \Theta_{(k_p+1 \ldots k_r)} \wedge \Theta_{0 \ldots 0}^{A_1} \wedge \Theta_{0 \ldots 0}^{A_2} + \\
&+ \sum_{s=0}^{r} \sum_{(k_s+1 \ldots k_r)} \Theta_{0 \ldots 0}^{i_s} \Theta_{0 \ldots 0}^{i_{s-1}q} \wedge \Theta_{0 \ldots 0}^{A_1} \wedge \Theta_{0 \ldots 0}^{A_2},
\end{align*}
\]

where

\[
\begin{align*}
\Theta_{0 \ldots 0}^{i_s} &\equiv c_{p}^z \Theta_{0 \ldots 0}^{i_s} \wedge \Theta_{0 \ldots 0}^{i_s} = 0 \\
\hat{\Theta}_{i_1 \ldots i_r}^{A_1} &\equiv \Theta_{i_1 \ldots i_r}^{A_1} \wedge \Theta_{0 \ldots 0}^{A_1} + \Theta_{0 \ldots 0}^{A_2} \wedge \hat{\Theta}_{i_1 \ldots i_r}^{A_2} = 0 \\
\text{denotes the summation over all sequences } (k_p + 1 \ldots k_r) \text{ such that } k_j = i_j \\
\text{or } k_j = 0, k_j = i_j - k_j \text{ and } \sum_{s=0}^{r} \text{ means the summation over the integers } s = p + 1, \ldots, r \text{ for which } i_s \neq 0.
\end{align*}
\]

Proof (by induction). For \( r = 1 \), (5) and (6) are equivalent to (4) and (4'). Assume by induction that the components of \( \Theta'_r \) satisfy (5) and (6). Then the forms \( \omega_{0 \ldots 0}^z, \omega_{i_1 \ldots i_r}^z \) of the dual basis to \( e_{0 \ldots 0}^{z_1}, \ldots, e_{i_1 \ldots i_r}^{z_1} \) satisfy the following structure equations of the group \( G' \):

\[
\begin{align*}
\text{d} \omega_{0 \ldots 0}^z &= \frac{1}{2} c_{p}^z \omega_{0 \ldots 0}^z \wedge \omega_{0 \ldots 0}^z, \\
\text{d} \omega_{i_1 \ldots i_p \ldots i_r}^z &= \sum_{(k_p + 1 \ldots k_r)} \omega_{0 \ldots 0}^{i_p} \omega_{(k_p+1 \ldots k_r)} \wedge \omega_{0 \ldots 0}^{A_1} \wedge \omega_{0 \ldots 0}^{A_2} + \\
&+ \sum_{s=0}^{r} \sum_{(k_s+1 \ldots k_r)} \omega_{0 \ldots 0}^{i_s} \omega_{0 \ldots 0}^{i_{s-1}q} \wedge \omega_{0 \ldots 0}^{A_1} \wedge \omega_{0 \ldots 0}^{A_2},
\end{align*}
\]

where

\[
\begin{align*}
\omega_{0 \ldots 0}^{i_s} &\equiv c_{p}^z \omega_{0 \ldots 0}^{i_s} \wedge \omega_{0 \ldots 0}^{i_s} = 0 \\
\omega_{0 \ldots 0}^{i_s} \wedge \omega_{0 \ldots 0}^{i_s} &\equiv 0.
\end{align*}
\]

Since \( \tilde{W}^{r+1} = W^1 (\tilde{W}^r) \), we can use (4). Hence the components of \( \Theta_{r+1} \) satisfy:

\[
\begin{align*}
\text{d} \Theta_{0 \ldots 0}^z &= \Theta_{0 \ldots 0}^z \wedge \Theta_{0 \ldots 0}^z_0, \\
\text{d} \Theta_{0 \ldots 0}^p &= \frac{1}{2} c_{p}^z \Theta_{0 \ldots 0}^z \wedge \Theta_{0 \ldots 0}^p + \Theta_{0 \ldots 0}^p \wedge \Theta_{0 \ldots 0}^{z_0} \\
\text{d} \Theta_{i_1 \ldots i_p \ldots i_r}^A &= \Omega, \\
\text{d} \Theta_{i_1 \ldots i_p \ldots i_r}^A = &\Theta_{i_1 \ldots i_r}^A \wedge \Theta_{0 \ldots 0}^{z_0},
\end{align*}
\]

where \( \Omega \) is the form which can be obtained formally from the form on the right side of (6) by adding zero to the end of every multiindex and

\[
\hat{\Theta}_{i_1 \ldots i_r}^A = \Theta_{i_1 \ldots i_r}^A.
\]
The exterior differentiation of (7) yields
\[
\Theta_q^g \wedge \{ \sum_{(k_p+1 \ldots k_r)} [\Theta_{0 \ldots 0 i_p(k_p+1 \ldots k_r)q} \wedge \Theta_{0 \ldots 0 B(k_p+1 \ldots k_r)0} + \\
+ \Theta_{0 \ldots 0 i_p(k_p+1 \ldots k_r)0} \wedge \Theta_{0 \ldots 0 B(k_p+1 \ldots k_r)q}] + \\
+ \sum_{s=p+1} \sum_{(k_s+1 \ldots k_r)} [\Theta'_{0 \ldots 0 i_s(k_s+1 \ldots k_r)q} \wedge \Theta'_{0 \ldots 0 i_p \ldots k_s-1 t(k_s+1 \ldots k_r)0} + \\
+ \Theta'_{0 \ldots 0 i_s(k_s+1 \ldots k_r)0} \wedge \Theta'_{0 \ldots 0 i_p \ldots k_s-1 t(k_s+1 \ldots k_r)q}] + \\
+ \Theta'_{0 \ldots 0 q} \wedge \Theta'_{0 \ldots i_s \ldots i_r} - d\Theta'_{0 \ldots i_s \ldots i_r} = 0.
\]

Using the generalized Cartan’s lemma, we get
\[
d\Theta'_{i_1 \ldots i_r t} = \sum_{(k_p+1 \ldots k_r+1)} [\Theta'_{0 \ldots 0 i_p(k_p+1 \ldots k_r+1) q} \wedge \Theta'_{0 \ldots 0 B(k_p+1 \ldots k_r+1)} + \\
+ \sum_{s=p+1} \sum_{(k_s+1 \ldots k_r+1)} [\Theta'_{0 \ldots 0 i_s(k_s+1 \ldots k_r+1) q} \wedge \Theta'_{0 \ldots 0 i_p \ldots k_s-1 t(k_s+1 \ldots k_r+1)} + \Theta'_{0 \ldots 0} \wedge \Theta'_{i_1 \ldots i_r+1 t},
\]

where \(i_{r+1} = q\) and \(\Theta'_{0 \ldots 0} \wedge \Theta'_{0 \ldots 0} \wedge \Theta'_{0 \ldots 0} = 0\). QED.

Remark 1. We have simultaneously proved that the equations (*) are the structure equations of \(G_n^1\).

4. The space \(W^1(P)\) is the first prolongation of \(P\); \(\tau(W^1(P)) = H^1(B) \otimes J^1(P)\). Denote by \(p_1, p_2\) the natural projections
\[
p_1 : H^1(B) \otimes J^1(P) \rightarrow H^1(B), \quad p_2 : H^1(B) \otimes J^1(P) \rightarrow J^1(P).
\]

Let \(\Psi_{\varphi, \sigma}\) be an allowable chart on the principal fibre bundle \(W^1(P) (B, G_n^1)\). The chart \(\Psi_{\varphi, \sigma}\) will be called a semi-holonomic allowable chart of the first order on \(P\) if
\[
j^1_{\varphi}[p_1 \tau(\sigma\varphi)] \in H^2(B), \quad j^1_{\varphi}[p_2 \tau(\sigma)] \in J^2(P).
\]

The set \(\overline{W}^2(P) = \overline{W}^1(W^1(P))\) of all 1-jets of all semi-holonomic allowable charts of the first order on \(P\) with the source \((0, e_1) \in R_n \times G_n^1\) will be called the second semi-holonomic prolongation of \(P\). We can identify
\[
\overline{W}^2(P) \equiv H^2(B) \otimes J^2(P).
\]

By induction, we define \(\overline{W}^r(P) = \overline{W}^1(\overline{W}^{r-1}(P))\) and call it, the \(r\)-th semi-holonomic prolongation of \(P\). It is possible to identify
\[
\overline{W}^r(P) \equiv \overline{H}^r(B) \otimes J^r(P).
\]

The space \(\overline{W}^r(P)\) is a principal fibre bundle \(\overline{W}^r(P) (B, \overline{G}_n^r)\), where the group \(\overline{G}_n^r\) can be identified with \(L_n^r \times T_n^r(G)\). This identification will be denoted by \(\xi\).
The canonical form \( \Theta_r \) on \( \bar{W}^r(P) \) is defined by (2), where \(*W* \) is a semi-holonomic admissible chart on \( \bar{W}^{r-1}(P) \). \( \Theta_r \) is a vector-valued form with values in \( \bar{w}_{n-1}^r(G) = T_{(0,\varepsilon_-,r-1)}(R^n \times \bar{G}_n^{\varepsilon_-,1}) \). In particular, we can identify \( B \equiv B \times \{e\} \), where \( \{e\} \) is the trivial one-element group. Then the canonical form \( \Theta_r \) on \( \bar{W}^r(B \times \{e\}) \) coincides with the canonical form of \( \bar{H}(B) \) introduced in [10].

Let \( \sigma \) be a global section of the principal fibre bundle \( \ast W^{r+1}(P) \) \( (\bar{W}^r(P), \ast G_n^{r+1}) = W^2(\bar{W}^{r-1}(P)) \cap \bar{W}^{r+1}(P) \) \), where \( \ast G_n^{r+1} \) denotes the kernel of the homomorphism \( j^1_2 : (\bar{G}_n^{\varepsilon_-})^2 \cap \bar{G}_n^{\varepsilon_+} \to \bar{G}_n^{\varepsilon_-} \). Then a form \( \Theta'_r \) (which will be called an admissible extension of \( \Theta_r \) on \( \bar{W}^r(P) \)) is defined analogously to (2'). \( \Theta'_r \) is a \( \bar{w}_{n}^r(G) \)-valued form. Obviously, the diagram

\[
\begin{array}{c}
T(\bar{W}^r(P)) \longrightarrow \bar{w}_{n}^r(G) \\
\downarrow (\tau^{-1}\xi)_* \quad \downarrow (\tau^{-1}\xi)_* \\
T(\bar{W}'(P)) \longrightarrow \bar{w}_{n}^r(G)
\end{array}
\]

is commutative. Diagram (8) implies the identification of \( \Theta_r \) on \( \bar{W}^r(P) \) with the restriction of \( \Theta_r \) on \( \bar{W}'(P) \) to \( (\tau^{-1}\xi)_* T(\bar{W}^r(P)) \). Let

\[
\Theta'_r(\sigma) = \Theta'_r \big| (\tau^{-1}\xi)_* T(\bar{W}^r(P)), \quad \Theta'^{A}_{1,...,r} = \Theta'^{A}_{1,...,r} \big| (\tau^{-1}\xi)_* T(\bar{W}^r(P)),
\]

where \( \Theta'^{A}_{1,...,r} \) are the components of \( \Theta'_r \) on \( \bar{W}^r(P) \). Consequently, writing \( \Theta'^{A}_{j_1,...,j_r} \) instead of \( \Theta'^{A}_{j_1,...,j_r} \) in equations (5) and (6) we obtain the structure equations of the canonical form \( \Theta_r \) on \( \bar{W}^r(P) \). Then the structure equations of \( \bar{G}_n^{\varepsilon_-} \) have the following form:

\[
d\omega^0_{0\ldots0} = \frac{1}{2}c^0_{\beta}\omega^\beta_{0\ldots0} \wedge \omega^0_{0\ldots0}
\]

\[
d\tilde{\omega}^A_{i_1\ldotsi_r} = \sum_{(k_r' + 1),...,(k_1')} \tilde{\omega}^B_{0\ldots0;k'_1\ldots;k_r'} \wedge \tilde{\omega}^A_{0\ldots0;k'_1\ldots;k_r'} + \sum_{p=0}^{r} \sum_{(k_r' + 1),...,(k_1')} \tilde{\omega}^A_{i_1\ldots0;k'_1\ldots;k_r'} \wedge \tilde{\omega}^A_{0\ldots0;i_1\ldots0;k'_1\ldots;k_r'},
\]

where

\[
\tilde{\omega}^A_{j_1\ldotsj_r} = \omega^A_{j_1\ldotsj_r} \big| (\tau^{-1}\xi)_* T(\bar{G}_n^{\varepsilon_-}),
\]

\[
\omega^A_{0\ldots0\beta_{i_1\ldotsi_r}} = c^\beta_{\beta'} \omega^\beta_{0\ldots0;i_1\ldotsi_r}, \quad \omega^A_{0\ldots0\beta_{i_1\ldotsi_r}} = 0.
\]

\( \Theta' \) is a vector-valued form with values in \( (\tau^{-1}\xi)_* \bar{w}_{n}^r(G) \equiv \bar{w}_{n}^r(G) \). Lemma 1 implies: \( \bar{w}_{n}^r(G) \) is determined by equations

\[
dc^A_{i_1\ldotsi_r} = dc^A_{k_1\ldotsk_r}, \quad \zeta(i_1 \ldots i_r) = \zeta(k_1 \ldots k_r).
\]

For this reason, if \( \zeta(i_1 \ldots i_r) = \zeta(k_1 \ldots k_r) \), then

\[
\tilde{\omega}^A_{i_1\ldotsi_r} = \tilde{\omega}^A_{k_1\ldotsk_r}, \quad \Theta'^A_{i_1\ldotsi_r} = \Theta'^A_{k_1\ldotsk_r}.
\]

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Therefore we can drop the zero components in all multiindices in (9). In particular,
considering $P = B \times \{e\}$, equations (9) for $A = q$ yield the structure equations of
the group $L_n'$. But these equations are identical with the equations (4,36) of (4).
Hence the structure equations of the non-holonomic differential group of order $r$
introduced by Lumiste in [5] coincide with the structure equations of $L_n'$.

5. Our previous considerations demonstrate sufficiently the importance of $\Theta_q'$ for
the determination of the structure equations of $\Theta_r$. Now, we shall show the role of
the holonomic prolongations of the second order in the definition of $\Theta_r'$. Let $\mathcal{G}^2_n$
be the kernel of the homomorphism $j_2^1 : \mathcal{G}^2_n \to \mathcal{G}^1_n$. It is easy to see that $\overline{W}^2(P)$ has
a structure of a principal fibre bundle $\overline{W}^2(P)(\overline{W}^1(P), \mathcal{G}^2_n)$. Let $\overline{\sigma}$ be a global section
of $\overline{W}^2(P)(\overline{W}^1(P), \mathcal{G}^2_n)$. Let $u \in W^1(P)$, $\overline{\sigma}(u) = j_{(0,\epsilon)}^1(P)^\Psi$, where $\Psi$ is a semi-holonomic
allowable chart of the first order on $P$. Let $X \in T_n(W^1(P))$, $X = j_0^1(\gamma(t))$. The form
\( \star \Theta_1(X) = j_0^1(\Psi^{-1}\gamma(t)) \)
will be called a semi-holonomic extension of $\Theta_1$. $\star \Theta_1$ is a vector-valued form on $W^1(P)$
with values in $w_n^1(G)$. Obviously, it holds
\[ (j_0^0) \star \Theta_1 = \Theta_1. \]
Since our following results are based on direct evaluations we shall consider directly
the bundles $W^1(R^n \times G)$ and $\overline{W}^2(R^n \times G)$. Let us identify
\[ W^1(R^n \times G) = R^n \times G^1_n = R^n \times (L^1_n \times T^1_n(G)), \]
\[ \overline{W}^2(R^n \times G) = R^n \times \mathcal{G}^2_n = R^n \times (L^2_n \times T^2_n(G)). \]
In what follows, we shall use on $W^1(R^n \times G)$ the local coordinates $x^i$, $a^i_q$, $b^a$, $b^a_q$
where
- $x^i$ are the canonical coordinates of $X \in R^n$,
- $a^i_q$ are the coordinates of a jet $a \in L^1_n$,
- $b^a$, $b^a_q$ are the coordinates of a jet $b \in T^1_n(G)$.
A non-holonomic allowable 1-chart $\Psi$ on $R^n \times G^1_n$ is given by $\Psi(x, h) = [\varphi(x), \sigma(x) h]$, where $\varphi(x)$ is a diffeomorphism $U \to V$, $\sigma(x)$ is a differentiable mapping
$U \to G^1_n$ ($U, V \subset R^n$ are open subsets, $0 \in U$) and $\sigma(X) h$ is the product in $G^1_n$. Let us
use the following notation: $x' = \varphi(x)$, $h = (a, b)$, $\sigma(x) = (1^a, 1^b) \in L^1_n \times T^1_n(G)$,
$(a', b') = \sigma(x) h$, $\Psi(x; a; b) = (x'; a'; b')$. Using (1) we obtain: If $\Psi$ is a semi-
holonomic allowable 1-chart on $R^n \times G^1_n$, $\Psi$ is given by the formula
\[ x'^{t} = 0 a^i_q x^q + 0 x^t, \]
\[ a' = 1 aa, \]
\[ b'^a = F^a(1^b a, b) = (1^b a)^a + b^a + \text{terms of higher order}, \]
\[ b'^{a}_q = F^{a}_q(1^b a, b) = (1^b a)^{a}_q + b^{a}_q + \text{terms of higher order}, \]
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where $F^a, F^b$ are functions determining the product in $G^1_n, 1a$ or $1b$ is the composition of jets in $L_n$ or $T_n^1(G)$ respectively and
\[
1a^i = a^i + a^i_{qp}x^p, \quad 1b^i = b^i + b^i_{pq}x^p, \quad 1b^a = b^a + b^a_{qp}x^p.
\]
Thus, the coordinates of $j^0_0 \Psi$ are $(0^a x^i; 0^a a^i, 0^a a^i_{qp}, 0^a b^i, 0^a b^i_{pq}, 0^a b^a_{qp})$. Let $e_p, e^p, e^q, e^a$ be the corresponding basis of $T_{(0, e)}(R^n \times G^1_n)$. Then
\[
\delta \Theta_1 = \Theta^p e_p + \Theta^q e^q + \Theta^ae_a + \Theta^a e^a,
\]
where $\Theta^p, \Theta^q, \Theta^a, \Theta^a$ are some scalar 1-forms on $W^1(R^n \times G^1_n)$. Denote by $E_p, E^q, E^a, E^a_q$ the basis of $T(W^1(R^n \times G^1_n))$ dual to $\Theta^p, \Theta^q, \Theta^a, \Theta^a_q$.

Let $\delta \Theta_1$ be determined by a section $\delta$. Let $\delta(u) = (0^a x^i, 0^a a^i, 0^a a^i_{qp}, 0^a b^i, 0^a b^i_{pq}, 0^a b^a_{qp})$. Let $\delta(u) = j_0^1 \Psi$. Then $\Psi$ is given by (9). Let $e_i = j_0^1 \gamma(v)$. $\gamma(v) = (x^p = \delta^p_i v; a^i_q = \delta^p_i; b^a = 0, b^a_q = 0).$ Using (9), we find directly:
\[
E_1(u) = j_0^1 \Psi(\gamma(v)) = 0^a a^i \frac{\partial}{\partial x^i} + 0^a a^i_{qp} \frac{\partial}{\partial a^i_p} + 0^a b^i \frac{\partial}{\partial b^a} + 0^a b^i_{pq} \frac{\partial}{\partial b^a_q}.
\]
and
\[
[E_1, E_q]_\omega = (0^a a^i - 0^a a^i_{qp}) \frac{\partial}{\partial x^i} + (0^a b^i - 0^a b^i_{pq}) \frac{\partial}{\partial x^i} + 0 \bmod E^p_A.
\]
But this implies:
\[
\delta \Theta_1 \left[ (0^a a^i_q - 0^a a^i_{qp}) \frac{\partial}{\partial x^i} + (0^a b^i_q - 0^a b^i_{pq}) \frac{\partial}{\partial x^i} \right] = 0 \bmod E^p_A
\]
if and only if $0^a a^i_q = 0^a a^i_{pq}, 0^a b^i_q = 0^a b^i_{pq}$.

Quite analogously to Kolář, [4], we evaluate
\[
[E_q, E^p] = -\delta^p_q E^p \bmod E^p_A, \quad [E_q, E^a] = -\delta^a_q E^a \bmod E^a_A,
\]
\[
[E_1, E_q] = 0 \bmod E^p_A.
\]
Now, calculating $d^\delta \Theta_1(X, Y)$, we deduce

**Proposition 2.** Let $\delta \Theta_1$ be a semi-holonomic extension of $\Theta_1$ determined by a section $\delta$. Then the value of $d^{\delta}_1$ lie in $W^2(P)$ if and only if
\[
d^{\delta}_1 = \Theta^p \wedge \Theta^p_1, \quad d^{\delta}_a = \frac{1}{2} \Theta^p \wedge \Theta^q \wedge \Theta^p \wedge \Theta^p_a.
\]

**Corollary.** In particular, if $P = B \times \{e\}$ then it holds: Let $\delta \Theta_1$ be a semi-holonomic extension of the canonical form $\Theta_1$ of $H^1(B)$ determined by a section $\delta$. Let $\Theta^p, \Theta^p_1$ be the components of $\delta \Theta_1$. Then the values of $\delta$ lie in $H^2(B)$ if and only if
\[
d^{\delta}_1 = \Theta^p \wedge \Theta^p_1.
\]
Remark 2 (due to Kolář). The preceding corollary is in the following relation to some properties of linear connections. The linear connection without torsion are in a one-to-one correspondence with the reductions of the principal fibre bundle $H^2(B)$ to the subgroup $L_n^1 < L_n^2$, see [3]. Libermann, [7], proves that the connections on $H^1(B)$ are in a one-to-one correspondence with the reductions of the principal fibre bundle $\tilde{H}^2(B)$ to $L_n^1$. We can explain this fact in the following way: It is

\[ \tilde{H}^2(B) = W^1(H^1(B)) = H^1(B) \otimes J^1(H^1(B)). \]

An element $(u, X) \in H^1(B) \otimes J^1(H^1(B))$ is semi-holonomic if and only if $u = j_i^0 X$. That is why we can identify

\[ \tilde{H}^2(B) = J^1(H^1(B)), \text{ see also [9].} \]

Let $\Gamma$ be a connection on $H^1(B)$ ($\Gamma$ is an invariant global section $H^1(B) \to J^1(H^1(B))$ see [5]). Hence $\Gamma$ determines a section $\tilde{T} : H^1(B) \to \tilde{H}^2(B)$. Denote by $R(\Gamma)$ the set $\tilde{T}(H^1(B))$. One can see easily that $R(\Gamma)$ is the reduction of $\tilde{H}^2(B)$ to $L_n^1$ treated by Libermann. In fact her considerations contain also the assertion that $\Gamma$ is without torsion if and only if $R(\Gamma) \subset H^2(B)$. We find it remarkable to show that this result follows also from the preceding corollary. Let $\Theta_2$ be the canonical form on $\tilde{H}^2(B)$. Denote by $\Theta_2$ its restriction to $R(\Gamma)$. Let $\omega$ be the canonical form of $\Gamma$ and let $\phi$ be the canonical form on $H^1(B)$. The diagram

\[
\begin{array}{ccc}
R^n & \xleftarrow{\phi} & T(H^1(B)) \\
\downarrow{pr_1} & & \uparrow \\
R^n \otimes L_n^1 & \xleftarrow{\tilde{\Theta}_2} & T(R(\Gamma)) \\
\downarrow{pr_2} & & \uparrow \\
L_n^1 & \xleftarrow{\omega} & T(H^1(B))
\end{array}
\]

(10)

is commutative ($L_n^1$ denotes the Lie algebra of $L_n^1$), see [6], Proposition 1. Let $\phi^p$ or $\omega_q^p$ be the components of $\phi$ or $\omega$ respectively. Then

\[ d\phi^p = \phi^q \wedge \omega_q^p + D\phi^p, \]

where $D$ denotes the absolute differential with respect to the connection $\Gamma$ and $D\phi$ is the torsion form of $\Gamma$. Diagram (10) implies that $\phi^i, \omega^i_j$ are the components of the semi-holonomic extension of $\omega$ determined by the section $\tilde{T}$. The preceding corollary yields: $D\phi^p = 0$ (e.i. the connection $\Gamma$ is without torsion) if and only if $R(\Gamma) \subset H^2(B)$. 661
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