Jacques Troué

On a theorem on the change of base point in a path connected space

_Czechoslovak Mathematical Journal_, Vol. 23 (1973), No. 1, 123–124

Persistent URL: http://dml.cz/dmlcz/101150

**Terms of use:**

© Institute of Mathematics AS CR, 1973

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project DML-CZ: The Czech Digital Mathematics Library http://dml.cz
In [1], Puppe has proved by elementary means the following

**Theorem 1.** If $A \subseteq X$, $B \subseteq Y$ are cofibrations, where $A$ and $B$ are closed subsets of $X$, $Y$ respectively, then also $(X \times B) \cup (A \times Y) \subseteq X \times Y$ is a cofibration.

There is no doubt that this general theorem will soon be part of any introduction to homotopy theory. The object of this note is to remark that it leads to an immediate proof of the following basic.
Theorem 2. (cf. [2] pp. 380—382). Let $X$ be a space with a non-degenerate base point $x_0$, and assume that $\{x_0\}$ is closed in $X$. Let $w$ be a path in the space $Y$. Then there is a bijection depending only on $[w]$, homotopy class of $w$

$$h_{[w]} : [X, x_0; Y, w(1)] \rightarrow [X, x_0; Y, w(0)],$$

where a representative of $h_{[w]}[f_1]$ is obtained by deforming $f_1$ so that the image of $x_0$ moves along $w$ from $w(1)$ to $w(0)$.

$[X, x_0; Y, w(i)]$ is the set of homotopy classes of maps $(X, x_0) \rightarrow (Y, w(i))$.

Proof. Recall (i) $x_0$ non-degenerate means $\{x_0\} \subset X$ is a cofibration, (ii) $I \subset I$ is a cofibration (obvious) and $1$ is closed in $I$.

Apply Theorem 1: $(x_0 \times I) \cup (X \times 1) \subset X \times I$ is a cofibration, i.e. each map $\{(x_0 \times I) \cup (X \times 1)\} \times I' \rightarrow Y$ can be extended to a map $(X \times I) \times I' \rightarrow Y$ with a given map $(X \times I) \times 1 \rightarrow Y$ coinciding with the first map where defined; or (cf. fig.) a map to $Y$ from the bottom — top — back faces and the section through $x_0$ can be extended to the whole cube.

Then the proof follows quickly:

A homotopy $f_1 \simeq f_1'$ in $(X, x_0; Y, w(1))$ is a map to $Y$ of the back face, a map $f_0$ corresponding to $f_1$ is obtained by filling in the top face, and similarly for the bottom face; a homotopy of a path in $Y$ rel $(w(0), w(1))$ is a map to $Y$ of the section through $x_0$. But then the Theorem of Puppe says that there is map from the front face to $Y$, i.e. $f_0 \simeq f_0'$ rel $x_0$. So $h_{[w]}$ is well-defined. Interchanging the role of the indices 0 and 1, shows that its inverse is also well-defined.

Remark. The condition that $\{x_0\}$ is closed in $X$ is necessary since Theorem 1 is false whithout it as a simple counter-example in [1] shows. This is not a severe restriction since already all points of a $T_1$-space are submitted to it.

References


Author's address: Department of Mathematics, McGill University, Montreal, Canada.