George M. Alexander; Charles W. Swartz

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LINEAR OPERATORS ON c_X

GEORGE ALEXANDER and CHARLES SWARTZ, Tucson

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In a recent paper [3], I. DOBRAKOV has posed several problems for spaces of vector-valued continuous functions on a locally compact Hausdorff space S. In this note we show that some results of J. BATT ([1]) on operator-valued series can be employed to solve two of these problems for the special case when S is the one-point compactification of the positive integers.

Before considering the problems, we introduce some notation. Let X, Y denote B-spaces and $L(X, Y)$ denote the B-space of all bounded linear operators from X into Y equipped with the usual operator norm. (Our terminology will be that of [4].) We denote by c_X the B-space of all X-valued, convergent sequences, $\{x_n\}$, equipped with the sup-norm, $\|\{x_n\}\| = \sup\{\|x_n\| : n \geqq 1\}$. Thus, c_X is the space $C_X(S)$ ([9]), where S is one-point compactification of the positive integers. Our results all pertain to linear operators on c_X.

The first problem which we consider is to give a characterization of unconditionally converging operators on c_X. (See [3], Theorem 3 and the remarks following.) Recall a bounded linear operator $T : X \to Y$ is said to be an unconditionally converging (u.c.) operator if T carries weakly unconditionally convergent (w.u.c.) series into unconditionally convergent (u.c.) series [9]. (A series $\sum x_n$ in X is w.u.c. if and only if the series $\sum |\langle x', x_n\rangle|$ converges for each $x' \in X'$ ([2], Lemma 2), and the series $\sum x_n$ is u.c. if and only if it converges unconditionally in norm ([7], [8]).) We first give necessary and sufficient conditions for a bounded linear operator $T : c_X \to Y$ to be u.c.

Recall ([6], p. 738) a bounded linear operator $T : c_X \to Y$ has a unique representation in the form

$$(1) \qquad \langle y', T\zeta\rangle = \langle y', T_0(\lim x_n)\rangle + \sum_{n=1}^{\infty} \langle y', T_n x_n\rangle$$

where $\zeta = \{x_n\} \in c_X$, $y' \in Y'$, $T_0 : X \to Y''$ and $T_n : X \to Y$ are bounded linear operators, and the series on the right-hand side of (1) satisfies $\sum_{n=1}^{\infty} \|y' T_n\| < \infty$. (See [1], Theorem 1 for some equivalent formulations regarding the condition on the series.)

Theorem 1. *A bounded linear operator* $T: c_X \to Y$ *is u.c. if and only if*

(i) *each* T_j *in* (1) *is u.c.* ($j = 0, 1, \ldots$)

(ii) *the series* $\sum T_j$ *is such that* $\sum T_j x_j$ *converges for each sequence* $\{x_j\} \subseteq X$, $\|x_j\| \leqq 1$. (See [1], Theorem 2 for equivalent formulations.)

Proof. First suppose that T is u.c. Define a continuous linear operator P_j ($j \geqq 1$) from X into c_X by $P_j x = \{\delta_{jn} x\}_{n=1}^{\infty}$. Let $\{x_j\} \subseteq X$ be such that $\|x_j\| \leqq 1$. Then $\sum P_j x_j$ is w.u.c. in c_X ([2], lemma 2.III) so that $\sum TP_j x_j = \sum T_j x_j$ is u.c. by (1). Hence (ii) is satisfied.

For $j \geqq 1$ and $x \in X$, equation (1) yields $TP_j x = T_j x$, so T_j is u.c. since T is u.c. ($j \geqq 1$). To establish (i) it remains to show T_0 is u.c.

Let $Q: X \to c_X$ be the continuous linear operator defined by $Qx = (x, x, \ldots)$. Now (ii) together with (1) shows $T_0 \in L(X, Y)$; and in fact, $T_0 x = TQx - \sum_{j=1}^{\infty} T_j x$ for all $x \in X$. Also (ii) and Theorem 2 of [1] show $\sum_{j=1}^{\infty} T_j x$ converges uniformly for all $\|x\| \leqq 1$, i.e., T_0 is the limit in the norm topology on $L(X, Y)$ of the sequence of u.c. operators $\{TQ - \sum_{j=1}^{n} T_j\}_{n=1}^{\infty}$. Thus T_0 is u.c. since the space of u.c. operators is closed in the norm of $L(X, Y)$.

Next suppose that T satisfies (i) and (ii). For each n, let $S_n : c_X \to Y$ be defined by $S_n \zeta = T_0(\lim x_n) + \sum_{j=1}^{n} T_j x_j$, $\zeta = \{x_n\}$. By (i) and (1), each S_n is u.c. By (ii) and Theorem 2 of [1], the series $\sum T_j x_j$ converges uniformly in Y for $\|x_j\| \leqq 1$ so that $S_n \to T$ in the norm of $L(c_X, Y)$. Since the space of u.c. operators is closed in the norm topology of $L(c_X, Y)$, T is u.c.

Remark. The necessity of conditions (i) and (ii) is established for the general case in [3], [10]; but the proof given here is simpler and involves no integration theory as in [3], [10].

In [1], Corollary 1, Batt has shown that $T: c_X \to Y$ is weak compact if and only if each T_j is weak compact and (ii) is satisfied. We recall that a B-space X has the property V if and only if for each B-space Y each u.c. operator $T: X \to Y$ is weak compact ([9]). By Theorem 1 and the result of Batt just quoted, we may obtain

Corollary 2. *X has the property V if and only if c_X has the property V.*

Proof. If X has V, c_X has V by the remarks above.

If c_X has V and $U: X \to Y$ is u.c., then $T: c_X \to Y$ defined by $T\zeta = Ux_1$, $\zeta = \{x_n\}$, is also u.c., and hence weak compact. Since for $x \in X$, $Ux = TP_1 x$, it follows that U is weak compact.

Remark. This result was established in [10] by other methods.

We finally show that c_X has the Dunford-Pettis property if and only if X has the Dunford-Pettis property. (See [3], §6, for a discussion of this problem and some partial results.) Recall that a B-space X is said to have the Dunford-Pettis property (D.P. property) if and only if for every B-space Y each weak compact operator $T: X \to Y$ carries weak Cauchy sequences into norm convergent sequences ([3], §6; [5]). For convenience we refer to a bounded linear operator $T: X \to Y$ which carries weak Cauchy sequences into norm convergent sequences as a completely continuous (cc) operator.

We first observe that the space of cc operators is closed in the uniform operator topology.

Proposition 3. *The space, $cc(X, Y)$, of all cc operators from X into Y is closed in the norm topology of $L(X, Y)$.*

Proof. Let $T_n \in \mathrm{cc}(X, Y)$, $T \in L(X, Y)$ and $\|T_n - T\| \to 0$. Let $\{x_n\}$ be weak Cauchy in X and $\varepsilon > 0$. There is $N > 0$ such that $\|T_N - T\| < \varepsilon/3B$, where $B > 0$ is such that $\|x_n\| \leqq B$. Now T_N is a cc operator so $\{T_N x_n\}_{n=1}^{\infty}$ is norm convergent, and there exists N_1 such that $n, m \geqq N_1$ implies $\|T_N(x_n - x_m)\| < \varepsilon/3$. Thus for $n, m \geqq N_1$, $\|T(x_n - x_m)\| \leqq \|(T - T_N) x_n\| + \|(T_N - T) x_m\| + \|T_N(x_n - x_m)\| < \varepsilon$. Hence, T is a cc operator.

The analogue of Theorem 1 for cc operators is

Theorem 4. *A bounded linear operator $T: c_X \to Y$ is a cc operator if and only if*

(a) *each T_j is a cc operator* $(j = 0, 1, \ldots)$
(b) *the series $\sum T_j$ satisfies condition* (ii) *of Theorem* 1.

Proof. First, suppose T is a cc operator. For $j \geqq 1$ denote by P_j the bounded linear operator from X into c_X defined by $P_j x = \{\delta_{nj} x\}_{n=1}^{\infty}$. Let $\{x_j\} \subseteq X$ be such that $\|x_j\| \leqq 1$, and set $z_n = \sum_{j=1}^{n} P_j x_j \in c_X$. Recall the dual of c_X can be identified with the space $l^1(X')$ of all absolutely summable X'-valued sequences ([6], p. 736; [3]), so that if $\alpha = \{x'_n\} \in c'_X$, then $\langle \alpha, z_n \rangle = \sum_{j=1}^{n} \langle x'_j, x_j \rangle \to \sum_{j=1}^{\infty} \langle x'_j, x_j \rangle$. Hence $\{z_n\}$ is weak Cauchy in c_X, and thus $\{Tz_n\} = \{\sum_{j=1}^{n} T_j x_j\}$ is norm convergent. That is, (b) is satisfied.

By a proof analogous to that used in Theorem 1 to show T a u.c. operator implies condition (i), it follows from the preceding paragraph and Proposition 3 that T a cc operator implies condition (a).

The converse is also established exactly as in Theorem 1. For each n set $S_n \zeta = T_0(\lim x_n) + \sum_{j=1}^{n} T_j x_j$, $\zeta = \{x_j\} \in c_X$. Each $S_n : c_X \to Y$ is a cc operator, and by (b) and Theorem 2 of [1], $\|S_n - T\| \to 0$. By Proposition 3, T is a cc operator.

Remark. It would be interesting to obtain an analogue of this characterization for general cc operators $T : C_X(S) \to Y$, when S is an arbitrary compact Hausdorff space.

Using this theorem we obtain

Corollary 5. *X has the D.P. property if and only if c_X has the D.P. property.*

Proof. Suppose X has the D.P. property and $T : c_X \to Y$ is weakly compact. Then by Corollary 1 of [1] and Theorem 4, T is a cc operator. Hence, c_X has the D.P. property.

Let c_X have the D.P. property and $U : X \to Y$ be weak compact. Then $T : c_X \to Y$ defined by $T\zeta = Ux_1$, $\zeta = \{x_n\}$, is weak compact, and hence T is a cc operator. For $x \in X$, $TP_1x = Ux$ so that U is a cc operator.

Remark. See Theorem 13a of [3] where the result is established by using integration theory. The proof given here is entirely elementary.

Corollary's 2 and 5 solve the two problems posed in [3] for the very special case when S is the one-point compactification of the positive integers. Of course, the methods employed here will not be applicable in the general case, but they do furnish some interesting applications of some of the results in [1] pertaining to operator-valued series.

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Author's addresses: George Alexander, New Mexico State University, Charles Swartz, University of Arizona, College of Liberal Arts, Department of Mathematics, Tucson, Arizona 85721, U.S.A.