Hana Petzelová
Periodic solutions of the equation $u_{tt} + u_{xxxx} = \varepsilon f(\cdot, \cdot, u, u_t)$


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PERIODIC SOLUTIONS OF THE EQUATION \( u_{tt} + u_{xxxx} = 8f(\cdot, \cdot, u, u_t) \)

HANA PETZELTOVÁ, Praha

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INTRODUCTION

The purpose of this paper is to prove the existence of \(2\pi\)-periodic solutions of the equation

\[
Lu \equiv u_{tt} + u_{xxxx} = 8f(\cdot, \cdot, u, u_t)
\]

\[
u(t, 0) = u_{xx}(t, 0) = u(t, \pi) = u_{xx}(t, \pi) = 0
\]

under the assumption that \(f\) is \(2\pi\)-periodic in \(t\).

The main point in the method of the proof is that the problem is solved in a Banach space \(A\), which can be decomposed into two complementary subspaces \(B\) and \(C\), where \(B\) is the null space of the operator \(L\) and \(L\) is boundedly invertible only on \(C\), the complement in \(A\) of \(B\). If we denote by \(P_1, P_2\) respectively the projectors of \(A\) onto \(B, C\) and seek the solution in the form \(u = v + w\), where \(v \in B, w \in C\), then the equation (0.1) is equivalent to the system

\[
P_1 F(v + w) = 0
\]

\[
Lw = 8 P_2 F(v + w)
\]

where \(F(u)(t, x) = f(t, x, u(t, x), u_t(t, x))\).

This method is used in several papers, e.g. in [1], [2], [3], [7], [8], [9], [10] to prove the existence of a solution to the equation (0.1) or to the wave equation. The essential assumption for solving the bifurcation equation (0.3) is

\[
f_{u_t} \geq \gamma > 0
\]

or

\[
f_u \geq \gamma > 0
\]
in the case that \( f \) depends only on \( t, x, u \). The assumption (0.6) is used for solving the bifurcation equation for the wave equation in [2], [3], [7], [9]. Hall [1] and Torelli [10] found weak solutions of the wave equation under a weaker monotonicity condition on \( f \), which permitted \( f_u = 0 \) to occur but at the expense of a growth condition on \( f \). In a later paper [8] Rabinowitz extended the results of [7]. He found classical solutions of the wave equation under the monotonicity condition which permitted \( f_u = 0 \) and which required no growth condition on \( f \). The existence of periodic solutions of a class of equations

\[
u_{tt} + (-1)^p \frac{\partial^{2p}}{\partial x^{2p}} u = ef(t, x, u)
\]

is proved in [1], [2].

In this paper the problem is solved in a slightly different way than in [2] which allows the function \( f \) to be dependent also on \( u_t \) under weaker conditions on the smoothness of \( f \).

The general sufficient conditions for the existence of periodic solutions to the equation \( Lu = g + ef(\cdot, \cdot, u, u_t, u_{xx}, u_t, \varepsilon) \) have been investigated by Krylová, Vejvoda [5]. Krylová [4] proved the existence of periodic solutions to the equation

\[
u_{tt} + A^2 u + cu + u_t + u_t|u_t| = f
\]

for \( n \)-dimensional Laplace operator.

In Section 1 some properties of the used spaces are established. The necessary and sufficient condition for the existence of a solution to the linear equation is given in Section 2. In Section 3 the nonlinear equation is treated. The special case of the equation (0.1) is investigated in Section 4. The case when \( f \) depends only on \( t, x, u \) is solved in Section 5.

1. PRELIMINARY

We begin with some notations. Let \( I = \langle 0, 2\pi \rangle \times \langle 0, \pi \rangle \), \( G = \mathbb{R} \times \langle 0, \pi \rangle \). Let \( D \) be the set of real valued, \( 2\pi \)-periodic, infinitely differentiable functions on \( G \) such that \( (\partial^{2k}/\partial x^{2k}) \phi(t, 0) = (\partial^{2k}/\partial x^{2k}) \phi(t, \pi) = 0, k = 0, 1, \ldots \) for \( \phi \in D \).

Denote by \( A_n \) the completion of \( D \) under the norm

\[
\|u\|_n = \left( \int_0^{2\pi} \int_0^{\pi} \left( \frac{\partial^n}{\partial t^n} u(t, x) \right)^2 + \left( \frac{\partial^{2n}}{\partial x^{2n}} u(t, x) \right)^2 \, dx \, dt \right)^{1/2}.
\]

\( A_n \) are Hilbert spaces with the inner product

\[
(u, v)_n = \left( \frac{\partial^n}{\partial t^n} u, \frac{\partial^n}{\partial t^n} v \right) + \left( \frac{\partial^{2n}}{\partial x^{2n}} u, \frac{\partial^{2n}}{\partial x^{2n}} v \right).
\]
where \((u, v) = \int_0^{2\pi} \int_0^{\pi} u(t, x) v(t, x) \, dx \, dt.\) For \(n = 0\) we shall simply write \(A, \| \cdot \|, (\cdot, \cdot).\) If \(k, l\) are integers, \(l > 0,\) define the functions \(e_{kl}\) by

\[
e_{kl}(t, x) = e^{ikt} \sin lx.
\]

The functions \(e_{kl}/\pi\) form a complete orthonormal system in the space \(A.\) Denote by \(\{u_{kl}\}\) the sequence of Fourier coefficients of the function \(u \in A.\)

\[
(1.2) \quad u_{kl} = (u, e_{kl})
\]

By means of integration by parts we get the following lemma.

**Lemma 1.1.** The functions \(e_{kl}^n = e_{kl}/\pi(k^{2n} + l^{4n})^{1/2}\) form a complete orthonormal system in \(A_n.\)

By Parseval’s equality and Lemma 1.1 we get using the integration by parts

\[
(1.4) \quad \pi^2 \| u \|_{n}^2 = \sum_{k=-\infty}^{\infty} \sum_{l=1}^{\infty} (k^2 + l^4) \| (u, e_{kl}) \|^2 = \sum_{k=-\infty}^{\infty} \sum_{l=1}^{\infty} (k^{2n} + l^{4n}) |u_{kl}|^2.
\]

This norm is equivalent to that used in [2] for \(n \geq 0\)

\[
\| u \|_{n}^2 = \sum_{k=-\infty}^{\infty} \sum_{l=1}^{\infty} (k^2 + l^4)^n |u_{kl}|^2,
\]

because

\[
k^{2n} + l^{4n} \leq (k^2 + l^4)^n \leq 2^{n-1}(k^{2n} + l^{4n}).
\]

In the sequel, the following two theorems from [2] will be used.

**Theorem 1.1.** If \(u \in A_k,\) then \(D_t^m D_x^n u\) are continuous when \(k \geq m + \frac{1}{2} n + 1\) and

\[
(1.5) \quad \| D_t^m D_x^n u \|_1 = \sup \| D_t^m D_x^n u(t, x) \| \leq \text{const} \| u \|_k.
\]

Let \(c_0\) denote the upper bound of the embedding operator \(A_1 \ni u \to u \in C^0,\) \(C^0\) being the space of continuous functions.

\[
(1.6) \quad \| u \|_\infty \leq c_0 \| u \|_1.
\]

Let \(f\) depend on \(t, x, u_1, u_2.\) For \(u \in A_1\) denote by \(F(u)\)

\[
(1.7) \quad F(u)(t, x) = f(t, x, u(t, x), u_1(t, x)).
\]

**Theorem 1.2.** Let \(u\) and \(v\) \(\in A_n\) for \(n \geq 2\) with \(\| u \|_n\) and \(\| v \|_n \leq b.\) Suppose \(f\) and its derivatives up to order \(2n\) are continuous and bounded whenever the arguments
$u_i$ are bounded. Then there are constants $c_1, c_2$ such that

$$
(1.8) \quad \|F(u)\|_{n-1} \leq c_1 \\
(1.9) \quad \|f_u(\cdot, \cdot, u, u_t) v_i\|_{n-1} \leq c_2 \|v\|_n, \quad i = 1, 2, \quad v_1 = v, \quad v_2 = v_t.
$$

Remark 1.1. To prove the condition (1.8) it suffices to suppose $2(n - 1)$ continuous derivatives of $f$.

Remark 1.2. The assertion (1.9) and the mean value theorem give immediately the relation

$$
(1.10) \quad \|F(u) - F(v)\|_{n-1} \leq 2c_2\|u - v\|_n.
$$

Lemma 1.2. The set $A^r_n = \{u \in A_n, \|u\|_n \leq r\}$ is a closed subset of $A$.

Proof. Let $\{u_j\}$ be a sequence of the elements of $A^r_n$ and let $u_j \to u$ in the space $A$. We prove that $u \in A^r_n$. By the Banach-Saks theorem there is a subsequence $\{v_j\}$ of arithmetic means of $\{u_j\}$ converging strongly to an element $v \in A^r_n$. But $u_j \to u$ in $A$. Thus $v_j \to u$ in $A$ and so $u = v$. Hence $u \in A^r_n$.

2. THE LINEAR EQUATION

We turn our attention to the existence in $A_n$ of solutions to the equation

$$
(2.1) \quad Lu = u_{tt} + u_{xxxx} = g \\
(2.2) \quad u(t, 0) = u_x(t, 0) = u(t, \pi) = u_{xx}(t, \pi) = 0.
$$

Definition. We say that $u \in A_n$ is a solution to the problem (2.1), (2.2) for $g \in A_n$, if $(u, L\phi)_n = (g, \phi)_n$ for all $\phi \in D$.

Remark 2.1. The choice of the space $D$ implies immediately that the boundary conditions (2.2) are fulfilled in the weak sense.

Let $u \in A_n$ be a solution of the equation

$$
(2.3) \quad Lu = 0.
$$

It is easily seen that $u$ has the representation

$$
(2.4) \quad u \sim \sum_{k^2 = 14} u_{kl} e_{kl}.
$$
Denote the set of such solutions by \( B_n \). The coefficients of an element in \( B_n \) are zero when \(|k| \neq n\) and hence for such \( u \) the norm reduces to

\[
|u|^2_n = \sum_{l=1}^{\infty} 2l^{4n} |u_{l}^{n+1}|^2 = 2 \int_0^{2\pi} \int_0^\pi \left| \frac{\partial^n}{\partial t^n} u(t, x) \right|^2 \, dx \, dt.
\]

In [2] the following lemma is proved:

**Lemma 2.1.** If \( u \in B_n \) for \( n \geq 1 \) and \( 2a_1 + a_2 \leq 2(n - 1) \), then \( D_i^n D_s^2 u \) is Hölder continuous with the exponent \( \frac{1}{2} \) and

\[
\|D_i^n D_s^2 u\|_{1/2} \leq \text{const} \|u\|_n
\]

where

\[
\|u\|_{1/2} = \sup \{ |u(t + h, x + k) - u(t, x)|, |h^2 + k^2|^{-1/4}, (t, x, h, k) \in I \}.
\]

Denote by \( C_n \) the complement of \( B_n \) in \( A_n \). Then the function \( u \in C_n \) has the representation

\[
 u \sim \sum_{k^2+l^4 \neq 14} u_{k} e_{kl}.
\]

Clearly, \((u, v)_n = 0\) for \( u \in B_n, v \in C_n \).

**Lemma 2.2.** The problem (2.1), (2.2) has a solution in \( A_{n+1} \) if and only if \( g \in C_n \). The solution is unique if its component in \( B_{n+1} \) is zero. The unique solution is given by \( u = Kg \) where \( K \) is defined by

\[
 K g(t, x) \sim \sum_{k^2+l^4 \neq 14} \frac{g_{kl}}{l^4 - k^2} e_{kl}(t, x).
\]

Finally, \( K : C_n \to C_{n+1} \) and

\[
\|Kg\|_{n+1} \leq \|g\|_n.
\]

**Proof.** For \( g \in C_n \) we have

\[
(Kg, L\varphi)_n = \lim_{N \to \infty} \left( \sum_{k=1}^N \sum_{l=1}^N \frac{g_{kl}}{l^4 - k^2} e_{kl}(t, x), L\varphi \right)_n = \lim_{N \to \infty} \left( \sum_{k=1}^N \sum_{l=1}^N g_{kl} e_{kl}, \varphi \right)_n = (g, \varphi)_n
\]

and so \( Kg \) is a solution of (2.1). On the other hand, if \( u \) is a solution of (2.1), then for \( \varphi \in D \cap B_{n} \) \((g, \varphi)_n = (u, L\varphi)_n = 0\). This implies that \( g \in C_n \). Let \( u \in C_n, v \in C_n \) be two solutions of (2.1). Then \( u - v \in C_n \) and \((u - v, L\varphi) = 0\) for all \( \varphi \in D \). Hence \( u - v \in B_n \). But \( C_n \cap B_n = \emptyset \). Hence \( u = v \). Further

\[
\|Kg\|_{n+1}^2 = \sum_{k^2+l^4 \neq 14} \left| g_{kl} \right|^2 \frac{k^{2(n+1)} + l^{4(n+1)}}{(l^4 - k^2)^2} \leq \max_{k^2+l^4 \neq 14} \frac{k^2 + l^4}{(l^4 - k^2)^2} \|g\|_n \|
\]
Since $|k| = t^2$, we have $(t^4 - k^2)^2 = (t^2 - k)^2 (t^2 + k)^2 \geq (t^2 + |k|)^2 = t^4 + k^2$.

Hence $(k^2 + t^4) (t^4 - k^2)^2 \leq 1$ and (2.8) holds.

We conclude this section with lemma which will be useful in the sequel.

**Lemma 2.3.** Let $\phi \in A$ and $\int_I \phi(t, x) \, dx \, dt = 0$. Then for $\psi \in A$ such that $m \leq \psi \leq M$ the following estimate holds.

\[
-\frac{1}{2} (M - m) \int_I |\phi(t, x)| \, dx \, dt \leq \int_I (\phi \psi)(t, x) \, dx \, dt \leq \frac{1}{2} (M - m) \int_I |\phi(t, x)| \, dx \, dt .
\]

**Proof.** Denote by $I_1 = \{(t, x) \in I, \phi(t, x) \geq 0\}$, $I_2 = I - I_1$. Then

\[
\int_I (\phi \psi)(t, x) \, dx \, dt = \int_{I_1} + \int_{I_2} \leq M \int_{I_1} \phi(t, x) \, dx \, dt + m \int_{I_2} \phi(t, x) \, dx \, dt =
\]

\[
= (M - m) \int_{I_1} \phi(t, x) \, dx \, dt + m \int_{I_2} \phi(t, x) \, dx \, dt = \frac{1}{2} (M - m) \int_I |\phi(t, x)| \, dx \, dt .
\]

On the other hand

\[
\int_I (\phi \psi)(t, x) \, dx \, dt = \int_{I_1} + \int_{I_2} \geq m \int_{I_1} \phi(t, x) \, dx \, dt + M \int_{I_2} \phi(t, x) \, dx \, dt =
\]

\[
= m \int_{I_1} \phi(t, x) \, dx \, dt + (M - m) \int_{I_2} \phi(t, x) \, dx \, dt =
\]

\[
= -\frac{1}{2} (M - m) \int_I |\phi(t, x)| \, dx \, dt .
\]

**Remark 2.2.** For $u \in B_1$ we get that $uu_t = \frac{1}{2} (\partial_t^2) u^2$ fulfills the assumptions of Lemma 2.3. In this case we have the estimate

\[
-\frac{1}{2} (M - m) \left\| u \right\| \left\| u_t \right\| \leq \int_I (\psi uu_t)(t, x) \, dx \, dt \leq \frac{1}{4} (M - m) \left\| u \right\| \left\| u_t \right\| .
\]

In the space $B_1$ we have $\left\| u \right\| \leq \left\| u_t \right\|$ and $\left\| u_t \right\| = \frac{1}{2} \left\| u \right\|$. Hence

\[
-\frac{1}{2} (M - m) \left\| u \right\|^2 \leq 2 \int_I (\psi uu_t)(t, x) \, dx \, dt \leq \frac{1}{4} (M - m) \left\| u \right\|^2 .
\]

This estimate will be used in the next section.

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3. THE NONLINEAR EQUATION

Now we turn to the problem

\[(3.1) \quad Lu = \varepsilon f(\cdot, \cdot, u, u_t)\]

with the boundary conditions (2.2).

Let \(P_1, P_2\) respectively denote the projectors of \(A\) onto \(B, C\). Then the equation (3.1) is equivalent to the system

\[(3.2) \quad \varepsilon P_1 F(v + w) = 0, \]
\[(3.3) \quad Lw = \varepsilon P_2 F(v + w)\]

where \(v = P_1 u\), \(w = P_2 u\). Let us denote

\[(3.4) \quad B_n^\varepsilon = \{u \in B_n, \|u\|_n \leq R\}, \quad R > 0, \]
\[(3.5) \quad C_n^r = \{u \in C_n, \|u\|_n \leq r\}, \quad r > 0.\]

**Theorem 3.1.** Let the function \(f\) have continuous derivatives up to the order \(2(n - 1)\), \(n \geq 2\), which are bounded when the argument is bounded. If the equation (3.2) has a unique solution \(v \in B_n^\varepsilon\) for each \(w \in C_n^r\) such that \(v(w)\) is Lipschitz continuous in \(w\) in the norm of the space \(A_1\), then the system (3.2), (3.3) has a unique solution in \(B_n^\varepsilon \times C_n^r\) for \(\varepsilon \neq 0\) and small.

**Proof.** By Lemma 2.2 we can write the equation (3.3) as

\[(3.5) \quad w = \varepsilon K P_2 F(v(w) + w).\]

The operator \(T \equiv K P_2 F(v(w) + w)\) is Lipschitz continuous and maps the set \(C_n^r\) into itself. If we choose \(\varepsilon \neq 0\) small enough, we get that \(\varepsilon T\) is a contraction in the norm of the space \(A_1\) of \(C_n^r\) into itself. Hence there is a fixed point of the operator \(\varepsilon T\) in \(C_n^r\) for each \(w \in C_n^r\) such that \(v(w)\) is Lipschitz continuous in \(w\) in the norm of the space \(A_1\), and by Lemma 1.2 \(C_n^r = C_n^r\). Thus there is \(w_0 \in C_n^r\) such that \(v(w_0) + w_0\) is a solution of the system (3.2), (3.3).

By Theorem 1.2 and (3.5) we get for \(w_0\)

\[(3.6) \quad \|w_0\|_n \leq \varepsilon c_1.\]

For \(w \in C_n^r\) we prove the existence of the solution of the equation (3.2) in \(B_n^\varepsilon\).

This will be done following Hall [2]. Let \(B(j), j = 1, 2, \ldots\) be the spaces

\[(3.7) \quad B(j) = \{u \in B, u(t, x) = \sum_{i=1}^J u_{\pm j, \pm \ell}(t, x)\} \]
equipped with the norm $\| \cdot \|_2$. Let $P^j_1$ be the projector from $A$ on $B(j)$ and $J$ be the operator on $B$ defined by

$$J u = \sum_{l=1}^{\infty} \frac{1}{l l_2} u_ \pm l e_ \pm l t.$$  

For $w \in C_n$ we define the operators

$$S^j_w v = J P^j_1 F(v + w),$$

$$S^j_w v = J P^j_1 F(v + w)$$

and we shall prove that there is such $v^j \in B(j)$ that

$$S^j_w v^j = 0.$$  

Then also $P^j_1 F(v + w) = 0.$

**Lemma 3.1.** Let $f$ be as in Theorem 3.1 and moreover let the following assumptions be fulfilled:

(i) $f_{u_1} \geq \gamma > 0$ on $G_1 = G \times (-c_0(R + r), c_0(R + r))^2,$

(ii) $\gamma - \frac{1}{2}(\sup_{G_1} f_u(t, x, u_1, u_2) - \inf_{G_1} f_u(t, x, u_1, u_2)) = \alpha > 0,$

(iii) $\sup_{G_1} |f_u(t, x, u_1, u_2)| < \alpha R,$

then the equations (3.10) have unique solutions in $B^R(j) = B(j) \cap B_2^R.$

**Proof.** For $v \in \partial B^R(j) = \{ u \in B(j), \| u \|_2 = R \}$ we have (as $\| v_r \| \leq R$ and by (2.10))

$$(S^j_w v, v)_2 = 2 \left( \frac{\partial}{\partial t} F(v + w), v_{tt} \right) =$$

$$= 2\left[ (f_{u_1} v_{tt}) + (f_u(v_1 + w_1), v_{tt}) + (f_u(v_1 + w_1, v_1), v_{tt}) \right] \geq$$

$$\geq R^2 \alpha - R \left[ \sup_{G_1} |f_u(t, x, u_1, u_2)| +$$

$$+ r \left( \sup_{G_1} |f_u(t, x, u_1, u_2)| \right) \right] \geq 0$$

if we choose $r$ small enough. Hence, by the known theorem (see e.g. [11]) there is such $v^j \in B^R(j)$, that (3.10) is fulfilled. As for the uniqueness, if $v_1, v_2 \in B^R(j)$ are two solutions of (3.10), then by the mean value theorem

$$0 = (S^j_w v_1 - S^j_w v_2, v_{1tt} - v_{2tt}) =$$

$$= -(F(v_1 + w) - F(v_2 + w), v_{1tt} - v_{2tt}) \leq - \frac{\alpha}{2} \| v_1 - v_2 \|_2^2.$$  

Hence $v_1 = v_2.$

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Lemma 3.2. Let $f$ and $v^j$ be as in Lemma 3.1. Then $\|v^j\|_n$ is bounded independently of $j$.

Proof. Lemma is proved for $n = 2$. For $n = 3$ we have

$$0 = \left( \frac{\partial^6}{\partial t^6} v^j, \frac{\partial^6}{\partial t^6} v^j \right) = -\left( \frac{\partial^2}{\partial t^2} F(v^j + w), \frac{\partial^3}{\partial t^3} v^j \right) =$$

$$= \left( f_{tt} + 2f_{tu}u^j + 2f_{tu}u_{tt} + f_{uu}u^j u^j + f_{uu}u^j u_{tt} + f_{uu}u_{tt} + f_{uu}u_{ttt} + f_{uuu}u^j, \frac{\partial}{\partial t} v^j \right)$$

where $u^j = v^j + w$. Since we assume the continuity of the second derivatives of $f$ and by Lemma 3.1 and Theorem 1.1 $\|u_{tt}\| \leq R$ and $\|u_t\|_{\infty} \leq c_0 R$, we have for $w \in C^3$

(3.11) $\gamma \|v_{tt}\| \leq \text{const} \left( 1 + \|v_{tt}\|^2 \right)$.

Rabinowitz has shown that if $\varphi \in D$, then

(3.12) $\|\varphi^2\| \leq a\delta^{1/4} \|\varphi\|_{1/2} \|\varphi_t\| + b(\delta)$

provided $\delta^{1/2} \|\varphi\|_{1/2} < \frac{1}{2}$. Here $a$ is a constant and $b$ depends on $\delta$, which itself can be chosen as small as needed. By Lemma 2.1 $\|v^j\|_{1/2} \leq c \|v^j\|_1 = c \|v^j\|_2$. Thus if $\delta$ is sufficiently small, (3.11), (3.12) combine to prove that $\|v^j\|_3 \leq \text{const}$. Estimates of $\|v^j\|_n$ for $n > 3$ are now quite evident.

Lemma 3.3. Let $f$ be as in Lemma 3.1. Then there is a unique element $v_0 \in B^R_n$ such that $S_wv_0 = 0$ for $n = 2, 3, \ldots$

Proof. By Lemma 3.1 and 3.2 there is a unique $v^j \in B(\tilde{j})$ for each $j = 1, 2, \ldots$ such that $S^j_wv^j = 0$. Further $\{v^j\}$ is bounded in $B_n$ for $n \geq 2$. By Lemma 2.1 and Theorem 1.1 the assumptions of Arzela's theorem for $v^j_t$ are fulfilled. Hence there is a subsequence, also denoted by $\{v^j\}$, and $v_0 \in A_1$ such that

(3.13) $\|v^j_t - v_{0t}\|_\infty \to 0$.

In the same way as in Lemma 1.2, with help of Banach-Saks theorem one proves that $v_0 \in B^R_n$.

$$S_wv_0 = S_wv_0 - S^j_wv_0 + S^j_wv_0 - S^j_wv^j \to 0$$

for $j \to \infty$, because $F(v^j + w) \to F(v_0 + w)$ when (3.13) holds.

Lemma 3.4. Let $f$ be as in Lemma 3.1. Then $v(w)$ is Lipschitz continuous in $w$ in the norm of the space $B_1$. 

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Proof. Let \(w_1, w_2 \in C^n_r\) and \(v_1, v_2 \in B^n_R\) be the corresponding solutions of the equation \(S_n v_i = 0\). Then

\[
0 = (S_n v_1 - S_n v_2, v_1 - v_2)_1 = 2(F(v_1 + w_1) - F(v_2 + w_2), v_1 - v_2) =
\]

\[
= 2[f(\cdot, \cdot, v_1 + w_1, v_1 + w_1) - f(\cdot, \cdot, v_2 + w_2, v_1 + w_1), v_1 - v_2] +
\]

\[
+ (f(\cdot, \cdot, v_2 + w_2, v_1 + w_1) - f(\cdot, \cdot, v_2 + w_2, v_2 + w_1), v_1 - v_2) +
\]

\[
+ (f(\cdot, \cdot, v_2 + w_2, v_2 + w_1) - f(\cdot, \cdot, v_2 + w_2, v_2 + w_2), v_1 - v_2)] =
\]

\[
= 2[(f_u(\text{int. pt.})(v_1 - v_2), v_1 - v_2) + (f_u(\text{int. pt.})(w_1 - w_2), v_1 - v_2) +
\]

\[
+ (f_u(\text{int. pt.})(v_1 - v_2), v_1 - v_2) + (f_u(\text{int. pt.})(w_1 - w_2), v_1 - v_2)] .
\]

By Lemma 2.3

\[
2(f_u(\text{int. pt.})(v_1 - v_2), v_1 - v_2) \geq -\frac{1}{2} (\sup_{G_1} f_u(t, x, u_1, u_2) - \inf_{G_1} f_u(t, x, u_1, u_2)) \|v_1 - v_2\|^2 .
\]

Thus

\[
\alpha \|v_1 - v_2\|^2 \leq \sup_{G_1} \left| f_u(t, x, u_1, u_2) \right| \|w_1 - w_2\| \|v_1 - v_2\| +
\]

\[
+ \sup_{G_1} \left| f_u(t, x, u_1, u_2) \right| \|w_1 - w_2\| \|v_1 - v_2\| .
\]

Hence

\[
\|v_1 - v_2\| \leq \frac{1}{\alpha} \left( \sup_{G_1} \left| f_u(t, x, u_1, u_2) \right| + \sup_{G_1} \left| f_u(t, x, u_1, u_2) \right| \right) \|w_1 - w_2\|. 
\]

We summarize our results in the next theorem.

**Theorem 3.2.** Let the function \(f\) fulfill the following assumptions. There are \(R > 0, r > 0\) such that

(i) \(f\) has continuous derivatives up to the order \(2(n - 1)\) on \(G_1 = G \times (-c_0(R + r), c_0(R + r))^2\), \(c_0\) is given by (1.6);

(ii) \(\frac{\partial^{2k}}{\partial x^{2k}} F(u)(t, 0) = \frac{\partial^{2k}}{\partial x^{2k}} F(u)(t, \pi) = 0, k = 0, 1, \ldots, n - 1, u \in A_n\);

(iii) \(f_u(t, x, u_1, u_2) \geq \gamma > 0\) on \(G_1\);

(iv) \(\gamma - \frac{1}{2} (\sup_{G_1} f_u(t, x, u_1, u_2) - \inf_{G_1} f_u(t, x, u_1, u_2)) = \alpha > 0\);

(v) \(\sup_{G_1} \left| f_u(t, x, u_1, u_2) \right| < \alpha R\).

Then if \(\varepsilon\) is sufficiently small, (3.1) has a unique \(2\pi\)-periodic solution in \(B^n_R \times C^n_r\) satisfying the conditions (2.2).
Remark 3.1. If \( f \) fulfills the assumptions of Theorem 3.2 for \( n = 3 \), then by Theorem 1.1 we get a classical solution of the problem.

Remark 3.2. Let \( f \) depend on \( t, x, u, u_t, u_x, u_{xx} \). Then if we substitute the condition (iv) in Theorem 3.2 by the condition
\[
\gamma - \frac{1}{2} \left( \sup_{G_2} f_u - \inf_{G_2} f_u \right) - \sup_{G_2} \left| f_{ux} \right| - \sup_{G_2} \left| f_{uxx} \right| = \alpha > 0
\]
where \( G_2 = G \times (-c_0(R + r), c_0(R + r)) \), we can prove in the same way the existence of a 2\( \pi \)-periodic solution of the equation
\[
Lu = ef(\cdot, \cdot, u, u_t, u_x, u_{xx})
\]
in the space \( A_2 \).

4. THE EQUATION \( Lu = e[a\nu + f(\cdot, \cdot, u)] \)

This problem will be solved in a slightly different way. No assumptions on the behavior of \( f_u \) will be needed. We shall prove the following lemma first.

**Lemma 4.1.** Let \( f \) be continuously differentiable up to the order \( 2(n - 1) \), \( n \geq 2 \). Then the equation
\[
w = \varepsilon KP_2 F(v + w) \quad (F(u) = au_t + f(\cdot, \cdot, u))
\]
has a unique solution \( w \in C^* \) for each \( v \in B^R_n \) such that
\[
\|w\|_n \leq \varepsilon c_1,
\]
\[
\|w(v_1) - w(v_2)\|_1 \leq \varepsilon K_1\|v_1 - v_2\|_1,
\]
\[
\|w\|_n \leq \frac{\varepsilon}{1 - \varepsilon x} (\|w\|_n + \|f(\cdot, \cdot, v + w)\|_{n-1}) \quad \text{for} \quad \varepsilon < \frac{1}{a}.
\]

**Proof.** The existence of the solution of (4.1) and the relation (4.2) is proved in the same way as in Theorem 3.1. (4.3) is established by means of the method of successive approximations and Theorem 1.2. Let \( v_1, v_2 \in B^R_1 \), \( u_0 = w(v_2), u_{n+1} = \varepsilon KP_2 F(v_1 + u) \). Then if \( K_2 < 1 \) is the Lipschitz constant of the operator \( \varepsilon T, Tu = KP_2 F(v_1 + u) \)
\[
\|w(v_1) - w(v_2)\|_1 = \lim_{j \to \infty} \|u_j - u_0\|_1 \leq \frac{1}{1 - K_2} \|u_1 - u_0\|_1 =
\]
\[
= \frac{1}{1 - K_2} \|\varepsilon KP_2 (F(v_1 + w(v_2)) - F(v_2 + w(v_2)))\|_1 \leq \varepsilon K_1\|v_1 - v_2\|_1 \]
\[
\leq \frac{\varepsilon}{1 - K_2} \|F(v_1 + w(v_2)) - F(v_2 + w(v_2))\| \leq \varepsilon K_1\|v_1 - v_2\|_1
\]

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This proves (4.4).

Now, it suffices to prove the existence of a solution to the equation $P_2F(v + w(v)) = F(v + w(v))$, or, which is the same, $P_1F(v + w(v)) = 0$. We shall investigate this problem again in the spaces $B(j)$ defined by (3.7), equipped in this case with the norm $\| \cdot \|_1$. As in Section 3, we define the operators $S^j$ on $B(j)$ by

$$S^j = JP^jF(v + w(v)), \quad Sv = JP^1F(v + w(v)).$$

Then for $v \in B^R(j)$

$$(S^jv, v)_1 = 2(\alpha(v_1 + w_1(v)) + f(\cdot, \cdot, v + w(v)), v_j) \geq \alpha \|v\|^2_1 - \|v\|_1(\alpha \epsilon_1 + \sup_{G_3} |f(t, x, u)|)$$

where $G_3 = G \times (-c_0(R + r), c_0(R + r))$, $c_0$ given by (1.6). If we suppose $\sup \{|f(t, x, u)|, (t, x, u) \in G_3\} < \alpha R$ and choose $\epsilon$ small enough, we get $(S^jv, v) \geq 0$ on $\partial B^R(j)$. Hence there is such $v^j \in B^R(j)$ that $S^jv^j = 0$. We proceed as in Section 3 and we show that $\|v^j\|_n \leq \text{const}$ independently of $j$. We have proved that $\|v^j\|_1 \leq R$. If $\|v^j\|_{n-1} \leq \text{const}$, then

$$0 = \left(F(v^j + w), \frac{\partial^{2n-1}}{\partial t^{2n-1}} v^j\right) = (-1)^{n-1} \left(\frac{\partial^{n-1}}{\partial t^{n-1}} F(v^j + w), \frac{\partial^n}{\partial t^n} v^j\right) =$$

$$= (-1)^{n-1} \left(\alpha \frac{\partial^n}{\partial t^n} v^j + \alpha \frac{\partial^n}{\partial t^n} w + \frac{\partial^{n-1}}{\partial t^{n-1}} f(\cdot, \cdot, v^j + w), \frac{\partial^n}{\partial t^n} v^j\right)$$

$$\|v^j\|_n \leq \|w\|_n + \frac{1}{\alpha} \|f(\cdot, \cdot, v^j + w)\|_{n-1} \Rightarrow \left(1 - \frac{\epsilon \alpha}{1 - \epsilon \alpha}\right) \|v^j\|_n \leq$$

$$\leq \text{const}$$

by Theorem 1.2. Hence $\|v^j\|_n \leq \text{const}$, if $\epsilon$ is small enough. In the same way as in section 3, with help of (4.3) one proves that for $n \geq 2$ there is a unique $v_0 \in B_n^R$ such that $Sv_0 = 0$.

**Theorem 4.1.** Let us suppose that there are $R > 0$, $r > 0$ such that $f$ is continuously differentiable up to the order $2(n - 1)$ on $G_3$,

$$\frac{\partial^{2k}}{\partial x^{2k}} F(u)(t, 0) = \frac{\partial^{2k}}{\partial x^{2k}} F(u) = (t, \pi) = 0, \quad k = 0, 1, \ldots, n - 1, \quad u \in A_n$$

and is $2\pi$-periodic in $t$. Further let $\sup \{|f(t, x, u)|, (t, x, u) \in G_3\} < \alpha R$. Then there is a unique solution of the problem $Lu = \varepsilon [xu + f(\cdot, \cdot, u)]$ with the boundary conditions (2.2) in $B_n^R \times C^R$, provided that $\varepsilon$ is small enough.
5. THE MORE DIMENSIONAL CASE

We shall treat the equation

\begin{equation}
L_k u = u_{tt} + A^2 u = e^t(-, -, u)
\end{equation}

for the \(k\)-dimensional Laplace operator, \(k = 1, 2, 3\), on the domain

\begin{equation}
Q_k = R \times Q_k, \quad \Omega_k = \langle 0, \pi \rangle^k
\end{equation}

with the boundary conditions

\begin{equation}
\left. u \right|_{\partial Q_k} = 0, \quad \frac{\partial^2 u}{\partial x_i^2} \bigg|_{x_i = 0} = 0, \quad i = 1, \ldots, k.
\end{equation}

Let \(A_{n,k}\) denote the completion of the set \(D_k\) of infinitely differentiable \(2\pi\)-periodic functions on \(Q_k\), such that

\[ \frac{\partial^{2m} u}{\partial x_i^{2m} |_{x_i = \pi}} = 0, \quad m = 0, 1, 2, \ldots \]

in the norms

\begin{equation}
\| u \|_{n,k} = \int_{Q_k} \left( \left| \frac{\partial^n u(t, x)}{\partial t^n} \right|^2 + \sum_{i=1}^k \left| \frac{\partial^{2n} u(t, x)}{\partial x_i^{2n}} \right|^2 \right) dx \, dt, \quad \bar{Q}_k = \langle 0, 2\pi \rangle \times \Omega_k, \quad x = (x_1, \ldots, x_k).
\end{equation}

\(A_{n,k}\) are Hilbert spaces with the inner products

\begin{equation}
(u, v)_{n,k} = \int_{Q_k} \left( \frac{\partial^n u(t, x)}{\partial t^n} \cdot \frac{\partial^n v(t, x)}{\partial t^n} + \sum_{i=1}^k \frac{\partial^{2n} u(t, x)}{\partial x_i^{2n}} \cdot \frac{\partial^{2n} v(t, x)}{\partial x_i^{2n}} \right) dx \, dt.
\end{equation}

In the same way as in Section 1 one proves that the functions

\begin{equation}
e^{k}_{rs}(t, x) = \pi^{k+1 \over 2k-1} \left( r^{2n} + \sum_{i=1}^k s_i 4^n \right)^{-1/2} e^{i r t} \sin s_1 x_1 \cdots \sin s_k x_k, \quad r \text{ integers },
\end{equation}

\(s = (s_1, \ldots, s_k), \quad s_i > 0 \) integers,

form a complete orthonormal systems in \(A_{n,k}\).

We say that \(u \in A_{n,k}\) is a solution to the problem \(L_k u = g\), \((5.3)\), \(g \in A_{n,k}\), if

\[(u, L\varphi)_{n,k} = (g, \varphi)_{n,k} \quad \text{for all } \varphi \in D_k.
\]

As in Section 3 we shall write \(A_{n,k} = B_{n,k} + C_{n,k}\), where \(B_{n,k}\) is the null space of the operator \(L_k\) and \(C_{n,k}\) is the orthogonal complement of \(B_{n,k}\) in \(A_{n,k}\). Clearly,

\begin{equation}
B_{n,k} = \{ u \in A_{n,k}, \quad u = \sum_{|r| = \sum s_i^2} u_{rs} e^{k}_{rs} \},
\end{equation}

\begin{equation}
u \in C_{n,k}, \quad u = \sum u_{rs} e^{k}_{rs} \Rightarrow u_{rs} = 0 \quad \text{for} \quad |r| = \sum s_i^2.
\end{equation}
The equation $L^u = g$ has a unique solution $u = Kg \in C_{n,k}$ for each $g \in C_{n,k}$, $g = \sum g_{rs} e^k_{rs}$ and for $Kg$ we have

$$Kg = \sum \frac{g_{rs}}{r^2 - (\sum s_i^2)^2} e^k_{rs},$$

(5.9)

$$\|Kg\|_{n,k} \leq \|g\|_{n,k}.$$  

(5.10)

Indeed, $|g_{rs}(r^2 - (\sum s_i^2)^2)^{-1}| \leq |g_{rs}|$ for $g \in C_{n,k}$.

**Lemma 5.1.** $u \in A_{n,k} \Rightarrow u \in C^0$ and

$$\|u\|_\infty \leq b_k \|u\|_{n,k} \quad \text{for} \quad n \geq 2.$$  

(5.11)

**Proof.** Let $u \in A_{n,k}$, $u = \sum u_{rs} e^k_{rs}$. Then

$$\sum |u_{rs}| = \sum (r^{2n} + \sum s_i^{4n})^{1/2} |u_{rs}| (r^{2n} + \sum s_i^{4n})^{-1/2} \leq \|u\|_{n,k}(\sum (r^{2n} + \sum s_i^{4n})^{-1})^{1/2} \leq b_k \|u\|_{n,k}$$

because $r^{2n} + \sum s_i^{4n} \geq r^n \prod s_i^{2n/k}$ and the series $\sum (r^n \prod s_i^{2n/k})^{-1}$ converges for $n \geq 2$, $k = 1, 2, 3$.

**Lemma 5.2.** $u \in B_{1,k} \Rightarrow u \in C^0$ and

$$\|u\|_\infty \leq d_k \|u\|_{1,k}.$$  

(5.12)

**Proof.** For functions from $B_{1,k}$ only the coefficients $u_{rs}$, where $|r| = \sum_{k} s_i^2$ are different from zero. Hence

$$\sum^k |u_{rs}| \leq \sum^k \frac{1}{r^2} \|u\|_{1,k}, \quad \sum^k \cdots = \sum_{s_i = 1}^{\infty} \cdots, \quad |r| = \sum s_i^2$$

$$\sum^k \frac{1}{r^2} = 2 \sum_{r=1}^{\infty} \frac{D_k(r)}{r^2},$$

where $D_k(r) = \sum 1$, $s_i$ integers. By RANDOL [13] the following estimate holds:

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\[ A_k(x) = C x^{k/2} + O(x^{k(k-1)/2(k+1)}) \] for \( k \geq 2 \), where \( A_k(x) = \sum_{\sum r_i \leq x} 1 \), \( s_i \) integers. Then

(5.13) \[ D_1(r) = A_k(r) - A_k(r - 1) = O(r^{k/2-1}) + O(r^{k(k-1)/2(k+1)}) \]

\[ D_2(r) = O(r^{1/3}) \quad D_3(r) = O(r^{3/4}) \]

Thus the series \( \sum k 1/r^2 \) converge for \( k = 1, 2, 3 \).

We solve the nonlinear equation under the assumptions

(5.14) \[ f(u) \geq \gamma > 0 \quad \text{on} \quad G_{4,k} = Q_k \times \langle -(d_k R + b_k r), \ d_k R + b_k r \rangle \]

for some \( R, \ r > 0 \)

(5.15) \[ \sup_{G_{4,k}} |f_i(t, x, u)| < \gamma R \]

in the same way as in Section 3. We prove that there is a unique \( v(w) \in B_{n,k} \) to each \( w \in \mathcal{C}_{n,k}, n \geq 2 \) such that \( P_i^k F(v + w) = 0 \) and this \( v \) is Lipschitz continuous in \( w \) in this case in the norm of the space \( A_{0,k} \), because \( f \) does not depend on the derivatives of \( u \). We work again in the spaces of finite dimension \( B_k(j) \)

(5.16) \[ B_k(j) = \{ u \in B_{0,k}, \ u = \sum_{|j| \leq j} u_{rs} e_{rs} \} \]

equipped with the norm \( \| \cdot \|_1,k \). We define the operators \( S^{i,k}_{w} = P_i^k F(v + w) \), where \( P_i^k \) is the projector of \( A_{0,k} \) on \( B_k(j) \) and under the assumptions (5.14), (5.15) and \( \varepsilon > 0 \) small enough we get \( (S^{i,k}_{w} v, v)_{1,k} \geq 0 \) on \( \partial B^{R}_{k}(j) = \{ u \in B_k(j), \| u \|_1,k = R \} \). With help of the following two lemmas we prove in the same way as in Section 3 the existence of such \( v_0 \in B^{k}_{n,k} \) that

(5.17) \[ S^{k}_{w} v_0 = 0 . \]

Lemma 5.3. If we define \( \omega(\delta) \) by

(5.18) \[ \omega(\delta) = \sup_{v \in B_{k,R}(j)} \sup_{(t, x) \in Q_n} \sup_{|h| \leq \delta} |u(t + h, x) - u(t, x)| \]

we get \( \omega(\delta) \to 0 \) for \( \delta \to 0 \).

Proof.

\[ |u(t + h, x) - u(t, x)| = \sum_{e_{rs}} u_{rs} \left( e^h_{rs}(t + h, x) - e^h_{rs}(t, x) \right) \leq \sum_{e_{rs}} |u_{rs}| \left| e^h_{rs} - 1 \right| \leq 2 \| u \|_{1,k} \sum_{r=1}^{\infty} \frac{D_r(r)}{r^2} \left| e^{r,h} - 1 \right| \leq 2 R \sum_{r=1}^{\infty} \frac{D_r(r)}{r^2} \left| e^{r,h} - 1 \right| . \]

The series converges uniformly and its value for \( h = 0 \) is zero.

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Lemma 5.4. Let \( u \) have two continuous derivatives. Then

\[
\int_{\Omega_k} v_t^4(t, x) \, dx \, dt \leq 3\omega(\delta) \int_{\Omega_k} v_{tt}^2(t, x) \, dx \, dt + C(\delta)
\]

where \( \omega(\delta) \) is given by (5.18) and \( \delta > 0 \) can be chosen as small as needed.

Remark. Integrating (5.19) over \( \Omega_k \) we get an estimate which we use instead of (3.12).

Proof is due to Rabinowitz [7]. Let \( \{\eta_i^2\} \) be a finite partition of the unity of \( \langle 0, 2\pi \rangle \) by \( 2\pi \)-periodic differentiable functions. Let the norm of the partition be \( \leq \delta \).

Integrating by parts and using the periodicity of \( \eta_i \) and \( v_i \), we have

\[
\int_0^{2\pi} \eta_i^2 v_t^4 \, dt = -3 \int_0^{2\pi} \eta_i^2 [v(x, t) - v(x, \tau_i)] v_t^2 v_{tt} \, dt - 2 \int_0^{2\pi} \eta_i \eta_{tt} [v(x, t) - v(x, \tau_i)] v_t^3 \, dt
\]

where \( \tau_i \in \text{supp} \eta_i^2 \). Thus

\[
\int_0^{2\pi} \eta_i^2 v_t^4 \, dt \leq 3\omega(\delta) \int_0^{2\pi} \left( |\eta_i^2 v_t v_{tt}| + |\eta_i \eta_{tt} v_t^3| \right) \, dt \leq 3\omega(\delta) \int_0^{2\pi} \left( \eta_i^2 v_t^4 + \eta_i^2 v_t^3 + \frac{(\eta_i \eta_{tt})^4}{4\pi^2} + \alpha^{4/3} v_t^4 \right) \, dt.
\]

Here we used the Hölder inequality and \( \alpha > 0 \) can be chosen arbitrarily. We sum over \( i \). We can assume \( \Sigma 1 \leq 2/\delta \). Taking \( \alpha = (\delta/2)^{3/4} \) we find

\[
\int_0^{2\pi} v_t^4 \, dt \leq 3\omega(\delta) \int_0^{2\pi} (v_t^4 + v_{tt}^3) \, dt + 3\omega(\delta) \left[ 2\delta^{-3} \int_0^{2\pi} (\eta_i \eta_{tt})^4 \, dt + \int_0^{2\pi} v_t^4 \, dt \right].
\]

Hence

\[
\int_0^{2\pi} v_t^4 \, dt (1 - 6\omega(\delta)) \leq 3\omega(\delta) \int_0^{2\pi} v_{tt}^2 \, dt + C(\delta)
\]

where

\[
C(\delta) = 6\omega(\delta) \delta^{-3} \sum_i^{2\pi} (\eta_i \eta_{tt})^4 \, dt.
\]

Now we prove that \( \| v(w_1) - v(w_2) \| \leq K \| w_1 - w_2 \| \) where \( v(w_i) \) are solutions of the equations (5.17) in \( B_{n,k}^R \) corresponding to \( w_i \in C_{n,k}^R \).

\[
0 = (S_{w_1} v_1 - S_{w_2} v_2, v_1 - v_2) = (F(v_1 + w_1) - F(v_1 + w_2) + F(v_1 + w_2) - F(v_2 + w_2), v_1 - v_2) = (f_a(\text{int. pt})(w_1 - w_2), v_1 - v_2) + (f_a(\text{int. pt})(v_1 - v_2), v_1 - v_2),
\]

\[
v_i = v(w_i).
\]
Thus
\[ \|v_1 - v_2\|^2 \leq \sup_{G_{4,k}} |f_\delta(t, x, u)| \|v_1 - v_2\| \|w_1 - w_2\| \]
\[ \|v_1 - v_2\| \leq \frac{1}{\gamma} \sup_{G_{4,k}} |f_\delta(t, x, u)| \|w_1 - w_2\|. \]

Thus the following theorem holds.

**Theorem 5.1.** Let \( n \geq 2 \) and assume that there are \( R > 0, r > 0 \) such that

(i) \( f \) is continuously differentiable up to the order \( 2n \) on \( G_{4,k} \)

\[ \frac{\partial^{2m}f(u)}{\partial x_i^{2m}} \bigg|_{x_i=0, x_i=\pi} = 0, \quad m = 0, 1, \ldots, n-1, i = 1, \ldots, k, u \in A_{n,k} \]

(ii) \( f_u \geq \gamma > 0 \) on \( G_{4,k} \),

(iii) \( \sup \{|f_\delta(t, x, u)|, (t, x, u) \in G_{4,k}\} < \gamma R \).

Then there is a unique solution to the problem (5.1), (5.3) in \( B_{n,k}^R \times C_{n,k}^r \) provided that \( \varepsilon > 0 \) is sufficiently small, \( k = 1, 2, 3 \).

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\* Author's address: 115 67 Praha 1, Žižná 25, ČSSR (Matematický ústav ČSAV v Praze).