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On transitive submanifolds of $\mathbb{C}^2$ and $\mathbb{C}^3$


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ON TRANSITIVE SUBMANIFOLDS OF $\mathbb{C}^2$ AND $\mathbb{C}^3$

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(Received June 12, 1972)

1. THE MAIN THEOREMS

In $\mathbb{C}^n$, consider the coordinates $(z_1, \ldots, z_n)$, $z_i = x_i + iy_i$. Let $\iota: \mathbb{C}^n \to \mathbb{R}^{2n}$ be the usual identification $\iota(z_1, \ldots, z_n) = (x_1, y_1, \ldots, x_n, y_n)$. In $\mathbb{R}^{2n}$, we have the well known induced endomorphism $I: \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, $I^2 = -\text{id}_n$, given by

$$I \frac{\partial}{\partial x^i} = \frac{\partial}{\partial y^i}, \quad I \frac{\partial}{\partial y^i} = -\frac{\partial}{\partial x^i}.$$ 

Denote by $\Gamma$ the pseudogroup of all local holomorphic diffeomorphisms in $\mathbb{C}^n$ (or $\iota(\Gamma)$ in $\mathbb{R}^{2n}$ resp.), let $\Gamma_s \subset \Gamma$ be the sub-pseudogroup of maps $z'_i = f_i(z_1, \ldots, z_n)$ satisfying

$$\det \frac{\partial(z'_1, \ldots, z'_n)}{\partial(z_1, \ldots, z_n)} = 1. \quad (1.1)$$

Let $M^m \subset \mathbb{C}^n$ be a real submanifold; let us write again $M^m$ instead of $\iota(M^m)$. Consider a point $p \in M^m$, the tangent space $T_p = T_p(M^m)$, and define $\tau_p$ as $T_p \cap JT_p$.

Lemma 1.1. Let $v_0 \in \tau_p$. In a neighbourhood $U \subset M^m$ of $p$, consider a vector field $v$ such that $v_p = v_0$ and $v_q \in \tau_q$ for each $q \in U$. The map $L_p^{(1)}: \tau_p \to T_p/\tau_p$ be given by $L_p^{(1)}(v_0) = \pi_1([v, Jv]_p)$, $\pi_1: T_p \to T_p/\tau_p$ being the projection; $L_p^{(1)}(v_0)$ depends on $v_0$ only. Let $\sigma_p \subset T_p$ be the linear hull of the set $\pi_1^{-1}(L_p^{(1)}(\tau_p))$. The map $L_p^{(2)}: \tau_p \to T_p/\sigma_p$ be defined by $L_p^{(2)}(v_0) = \pi_2([v, [v, Jv]]_p)$, $\pi_2: T_p \to T_p/\sigma_p$ being the projection; $L_p^{(2)}(v_0)$ depends on $v_0$ only.

$L_p^{(1)}$ and $L_p^{(2)}$ are the so-called Levi maps.

Write $G(M^m) = \{ \gamma \in \Gamma; \gamma(M^m) = M^m \}$ and $G_s(M^m) = G(M^m) \cap \Gamma_s$. We propose to prove the following two theorems.

306
Theorem 1.1. Consider the case \( n = 2, m = 3, \) i.e., \( M^3 \subset \mathbb{C}^2. \) Suppose \( L_p^{(1)} \neq 0 \) at each point \( p \in M^3. \) If \( G_s(M^3) \) is transitive on \( M^3, \) then it is a Lie group with \( \dim G_s(M^3) \leq 4. \) Consider the following manifolds

\[
\begin{align*}
N^3 & : z_1 \bar{z}_1 + z_2 \bar{z}_2 = r^2 \quad (r > 0), \\
N^3_R & : z_1 \bar{z}_2 + \bar{z}_1 z_2 = 2R \quad (R > 0), \\
N^3_0 & : i(z_2 - \bar{z}_2) = (z_1 - \bar{z}_1)^2 .
\end{align*}
\]

Let \( \dim G_s(M^3) = 4. \) Then there is exactly one manifold among the manifolds \( N^3, N^3_R, N^3_0 - \) denote it by \( N^3 - \) with the following property: choose \( p \in M^3, q \in N^3, \) then there is a neighbourhood \( U \subset M^3 \) of \( p \) and a \( \gamma \in \Gamma \) such that \( \gamma(p) = q, \gamma(U) \subset N^3. \) The groups \( G_s(N^3), G_s(N^3_R), G_s(N^3_0) \) are given by

\[
\begin{align*}
z'_1 & = \alpha z_1 - \beta z_2 , \quad z'_2 = e^{i\alpha} (\beta z_1 + \alpha z_2), \\
z'_1 & = e^{i\alpha} (az_1 + ibz_2) , \quad z'_2 = e^{i\alpha} (icz_1 + dz_2), \\
z'_1 & = e^{i\alpha} (az_1 + b + ci) , \\
z'_2 & = 4e^{i\alpha} cz_1 + i(1 - e^{2i\alpha}) z_1^2 + z_2 + d + 2c^2 i ,
\end{align*}
\]

where \( \alpha, \beta \in \mathbb{C}, \alpha \beta + \beta \beta = 1, \alpha \in \mathbb{R}; \)

where \( a, b, c, d, f \in \mathbb{R}, ad + bc = 1; \)

where \( a, b, c, d, f \in \mathbb{R} . \)

Theorem 1.2. Consider the case \( n = 3, m = 4, \) i.e., \( M^4 \subset \mathbb{C}^3. \) Suppose \( \dim \tau_p = 2, L_p^{(1)} \neq 0, L_p^{(2)} \neq 0 \) at each point \( p \in M^4. \) If \( G(M^4) \) is transitive on \( M^4, \) \( G(M^4) \) is a Lie group and \( \dim G(M^4) \leq 5. \) Let us consider a manifold \( M^4 \) with \( \dim G(M^4) = 5 \) and the manifold \( N^4 \) given by

\[
\begin{align*}
\bar{z}_2 - z_2 & = i(\bar{z}_1 - z_1)^2 , \quad \bar{z}_3 - z_3 = (\bar{z}_1 - z_1)^3 .
\end{align*}
\]

If \( p \in M^4, q \in N^4, \) are arbitrary points, there is a neighbourhood \( U \subset M^4 \) of \( p \) and a \( \gamma \in \Gamma \) such that \( \gamma(U) \subset N^4, \gamma(p) = q, \) i.e., \( M^4 \) and \( N^4 \) are locally \( \Gamma \)-equivalent. The group \( G(N^4) \) is

\[
\begin{align*}
z'_1 & = az_1 + b + ci , \\
z'_2 & = 4acz_1 + a^2 z_2 + d + 2c^2 i , \\
z'_3 & = -12ac^2 z_1 - 6a^2 cz_2 + a^3 z_3 + f - 4c^3 i ,
\end{align*}
\]

where \( a, b, c, d, f \in \mathbb{R} . \)

307
The first two chapters of this paper are devoted to the equivalence problems. The treatment is based on the theory of partial differential equations due to K. KURANISHI's notes *Lectures on involutive systems of partial differential equations* (Publ. da Soc. Math. de Sao Paulo, 1967) which are unfortunately not very well known. Chapter 3 contains the theory of structures induced on manifolds $M^3 \subset \mathcal{C}^2$ with respect to the pseudogroup $\Gamma_s$ and the proof of Theorem 1.1; in chapter 4, the manifolds $M^3 \subset \mathcal{C}^2$ with $\dim G_s(M^3) = 3$ are studied. Finally, Theorem 1.2 is proved in the last chapter.

From the literature, I mention just two papers of E. CARTAN devoted to the determination of all manifolds $M^3 \subset \mathcal{C}^2$ with $\dim G(M^3) \geq 3$ (Annali di Mat., 11, 1932, 17–90 and Verh. int. math. Kongr. Zürich, t. II, 1932, 54–56). Unfortunately, these papers are written in such a way that I do not fully understand them.

Parts of the results have been obtained during my stays in the USSR (State Univ. at Vilnius) and India (Delhi, Punjab and Bombay Univ. and Tata Inst. of Fund. Research). The paper has been written during my stay at the Humboldt-Univ. in Berlin (GDR). My thanks go to all these institutions.

2. SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

A fiber manifold is a triple $(M, N, \varrho)$, where:

(i) $M$ and $N$ are analytic manifolds, $\dim M = n + m$, $\dim N = n$;
(ii) $\varrho : M \rightarrow N$ is an analytic map of $M$ onto $N$;
(iii) to each $y \in M$ there exists its coordinate neighbourhood $U \subset M$ such that

$$
\begin{array}{ccc}
U & \xrightarrow{\mu_1} & \mathbb{R}^n \times \mathbb{R}^m \\
\varrho & \downarrow & \\
\varrho(U) & \xrightarrow{\mu_2} & \mathbb{R}^m
\end{array}
$$

is commutative; here, $(U, \mu_1)$ and $(\varrho(U), \mu_2)$ are charts and $pr_1$ is the natural projection. Denote by $J^k = J^k(M, N, \varrho)$ the analytic manifold of all $k$-jets of local sections of the fiber manifold $(M, N, \varrho)$; let us write $J^0 = M$, $J^{-1} = N$. The triple $(J^l, J^k, \varrho^l_k)$, $l > k$, is again a fiber manifold, $\varrho^l_k$ being the natural projection.

Let $X \in J^k$, $y = \varrho^0_k(X) \in M$, $x = \varrho^{-1}_k(X) \in N$. The space $Q_x(J^k) \subset T_x(J^k)$ be defined by the exact sequence

$$0 \rightarrow Q_x(J^k) \rightarrow T_x(J^k) \xrightarrow{d\varrho^k_{k-1}} T_{\varrho^k_{k-1}}(J^{k-1}) \rightarrow 0.$$

Let $\xi \in T_x(J^k)$. Then there exists a neighbourhood $U \subset M$ of $y$ and local sections $f(t) : \varrho(U) \rightarrow M$, $t \in (-\varepsilon, \varepsilon)$, such that

$$\xi = \frac{d}{dt} j^k_x(f(t)) \bigg|_{t=0}.$$
The section \( f(t) \) be chosen in such a manner that \( j^{-1}_x(f(t)) = \varphi_{k-1}(X) \). Let \( U \) be such that we have local coordinates \((x^i, y^a); i = 1, \ldots, n; a = 1, \ldots, m; \) in it, \((y^a)\) be the local coordinates in \( \varrho(U) \). In \( \varrho(U) \), the local section \( f(t) \) be given by \( y^a = f^a(x^i, t) \). The mapping

\[
\tau : Q_x(J^k) \to Q_y(M) \otimes S^k T^*_x(N)
\]

be defined by

\[
\tau(\xi) = \frac{\partial^{k+1} f^a(x^i, 0)}{\partial t \partial x^{i_1} \cdots \partial x^{i_k}} \otimes dx^{i_1} \otimes \cdots \otimes dx^{i_k},
\]

here, \((x^i_d)\) are the coordinates of the point \( x \) and \( S^kV \) is the \( k \)-th symmetric tensor product of the space \( V \). The mapping \( \tau \) does not depend on the choice of coordinates \((x^i, y^a)\). It is an isomorphism which is called the fundamental identification.

Let \( R^k \subset J^k \) be a submanifold. \( R^k \) is said to be regular at the point \( X \in R^k \) if there is a neighbourhood \( V \subset J^k \) of \( X \) and functions \( f_a : V \to \mathcal{R}; a = 1, \ldots, A; \) with the following properties:

(i) \( A + \dim R^k = \dim J^k \),

(ii) \( V \cap R^k = \{ Y \in J^k; f_a(Y) = 0 \text{ for } a = 1, \ldots, A \} \),

(iii) \( df_1, \ldots, df_A \in T^*_x(J^k) \) are linearly independent.

A submanifold \( R^k \subset J^k \) is said to be a partial differential equation of order \( k \) if it is regular at each of its points. The section \( f : U \to M, U \subset N \) being an open set, is said to be a solution of \( R^k \) if \( f(f(t)) \in R^k \) for each \( t \in U \).

Be given a function \( F : V \to \mathcal{R}, V \subset J^k \) being an open set; further, let \( v \) be a vector field on \( \varrho^{-1}(V) \subset N \). The function \( \partial_v F : (\varrho^{-1}(V))^{-1}(V) \to \mathcal{R} \) be defined as follows. Let \( X \in (\varrho^{-1}(V))^{-1}(V) \), and let \( f : N \to M \) be a local section such that \( X = j_{x_0}^1(f) \), \( x_0 = \varrho^{-1}(X) \). Consider the local section \( j^k f : N \to J^k \). Then we have the local map \( F \circ j^k f : N \to \mathcal{R} \); set \( (\partial_v F)(X) = v(F \circ j^k f)|_{x=x_0} \).

The differential equation \( R^k \) being given in a neighbourhood \( V \subset J^k \) of its point \( X \in R^k \) as \( \{ X \in V; f_a(X) = 0 \text{ for } a = 1, \ldots, A \} \), define \( pR^k|_{(\varrho^{-1}(V))^{-1}(V)} = \{ X \in (\varrho^{-1}(V))^{-1}(V) \subset J^{k+1}; f_a(\varrho^{-1}(X)) = 0 \text{ for } a = 1, \ldots, A \} \) and for each vector field \( v \in T(\varrho^{-1}(V)) \). It is easy to see that this definition does not depend on the choice of the neighbourhoods \( V \) and the functions \( f_a \); thus we have a well defined subset \( pR^k \subset J^{k+1} \) which is called the prolongation of \( R^k \).

Let \( R^k \) be a differential equation, \( X \in R^k \); \( R^k \) be given — in a neighbourhood of the point \( X \) — by means of the functions \( f_a \). Set

\[
C_X(R^k) = \{ \xi \in Q_x(J^k); \xi f_a = 0 \text{ for } a = 1, \ldots, A \}.
\]

By means of the fundamental identification and the natural mappings, we get

\[
C_X(R^k) \subset Q_x(J^k) = Q_y(M) \otimes S^k T^*_x(N) \subset Q_y(M) \otimes S^{k-1} T^*_x(N) \otimes T^*_x(N) = Q_{\varphi_{k-1}(x)}(J^{k-1}) \otimes T^*_x(N);
\]

309
here, \( y = \varphi_k(x), x = \varphi_{k-1}(X) \). Write

\[
A = C_X(R^k), \quad F = Q_{\varphi_{k-1}}(J^{k-1}), \quad E = T_x(N),
\]

and define

\[
pA = (A \otimes E^*) \cap (F \otimes S^2 E^*).
\]

Let \( e_1, \ldots, e_n \) be a basis of \( E \), let \( e^1, \ldots, e^n \) be the dual basis. Let \( E^*_{n-1} \subseteq E^* \) be the subspace spanned by the vectors \( e^{i+1}, \ldots, e^r \). Set

\[
(2.1) \quad A(i) = A \cap (F \otimes E^*_{n-1}), \quad \tau_i = \dim A(i).
\]

The basis \( e_1, \ldots, e_n \) is called quasi-regular with respect to \( A \) if

\[
(2.2) \quad \dim pA = \tau_0 + \cdots + \tau_{n-1};
\]

the space \( A \) is said to be involutive if there is a basis which is quasi-regular with respect to it.

**Definition 2.1.** The differential equation \( R^k \) is called involutive at the point \( X \in R^k \) if: (1) there is a neighbourhood \( V \subseteq J^k \) of \( X \) such that \( pR|_V, V_1 = (\varphi_{k+1})^{-1}(V) \), is a submanifold of \( J^{k+1} \) and \( (pR^k|_V, R^k|_V, \varphi_{k+1}) \) is a fiber manifold; (2) the space \( C_X(R^k) \) is involutive. \( R^k \) is involutive if it is involutive at each point \( X \in R^k \).

**Theorem 2.1.** Let the differential equation \( R^k \) be involutive at \( X_0 \in R^k \). Suppose that in a neighbourhood of the point \( x_0 = \varphi_{k-1}(X_0) \in N \) we have local coordinates \((x^1, \ldots, x^s)\) such that \( \partial/\partial x^1|x_0, \ldots, \partial/\partial x^s|x_0 \) is a quasi-regular basis. Then there is a neighbourhood \( V \subseteq J^k \) of the point \( X_0 \) such that, for each \( X \in R^k \cap V, \partial/\partial x^1|_X, \ldots, \partial/\partial x^s|_X; x = \varphi_{k-1}(X) \); is a quasi-regular basis with respect to \( C_X(R^k) \).

Consider again the subspace \( A = F \otimes E^* \). Let \( e \in E \); the linear map \( \delta_e : E^* \to \mathbb{R} \) by defined by \( \delta_e(e^*) = e^*(e) \) for \( e^* \in E^* \). \( W \) being a vector space, define the linear map \( \delta_e : W \otimes E^* \to W \) by means of \( \delta_e(w \otimes e^*) = e^*(e) w \). For each vector \( e \in E \), we have thus a linear map \( \delta_e : pA \to A \), this map being the restriction of \( \delta_e : A \otimes E^* \to A \).

**Theorem 2.2.** Let \( A = F \otimes E^* \) and let \( e_1, \ldots, e_n \) be a basis of \( E \). Consider the maps

\[
(2.3) \quad \delta_{e_{i+1}} : pA(i) \to A(i); \quad i = 0, \ldots, n - 1.
\]

The basis \( e_1, \ldots, e_n \) is quasi-regular if and only if the maps (2.3) are onto.

Be given a fiber manifold \((M, N, \varphi)\) and a submanifold \( N_1 \subseteq N \), \( \dim N = \dim N_1 + 1 \). Set \( J^k_1 = (\varphi_{k-1})^{-1}(N_1) \subseteq J^k(E); (J^k_1, N_1, \varphi_{k-1}) \) is again a fiber manifold. Consider the maps \( \sigma^k : J^k_1 \to J^k(M_1, N_1, \varphi); M_1 = \varphi^{-1}(N_1) \); defined as follows. Let \( X \in J^k_1, \varphi_{k-1}(X) = x \in N_1 \). Then there is a local section \( f : N \to M \) such that
Theorem 2.3. Let $R^1 \subset J^1(M, N, \varrho)$ be a differential equation of order one, suppose that $R^1$ is involutive at $X \in R^1$. Let $N_1 \subset N$ be a submanifold such that:

(i) $x = c^{1}_{-1}(X) \in N_1$, (ii) dim $N = \dim N_1 + 1$, (iii) there is a quasi-regular basis $e_1, \ldots, e_n \in T_x(N)$ with respect to $C_x(R^1)$ such that $e_1, \ldots, e_{n-1} \in T_x(N_1)$. Then $S^1 \subset J^1(M_1, N_1, \varrho)$ is a differential equation involutive at $X$. Let $\sigma^1 : N_1 \to M_1$ be a solution of $S^1$ defined in a neighbourhood of $x \in N_1$. Then there exists a neighbourhood $U \subset N$ of the point $x$ and a solution $\sigma : U \to M$ of the equation $R_1$ such that $\sigma|_{N_1 \cap U} = \sigma^1|_{N_1 \cap U}$.

3. INDUCED STRUCTURES

Let us consider the space $\mathbb{G}^n$ and its coordinates $z_i = x_i + i x_{n+i}; i = 1, \ldots, n$. Let $\mathbb{R}^{2n}$ be the real representation of $\mathbb{G}^n$ with the coordinates $(x_i, x_{n+i})$ endowed with the automorphism $I : \mathbb{R}^{2n} \to \mathbb{R}^{2n}, I^2 = -\text{id.}$, given by

\[
(I \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_{n+i}}, \quad I \frac{\partial}{\partial x_{n+i}} = -\frac{\partial}{\partial x_i}; \quad i = 1, \ldots, n. \]

Further, consider the fiber manifold $E = (\mathbb{R}^{2n} \times \mathbb{R}^{2n}, \mathbb{R}^{2n}, \pi_1), \pi_1 : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ being the natural projection onto the first factor. In $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$, we have the coordinates $(x_i, x_{n+i}, y_i, y_{n+i})$; the coordinates of the prolongation $J^1(E)$ are $(x_i, x_{n+i}, y_i, y_{n+i}, y_{i,j}, y_{i,n+i}, y_{n+i,j}, y_{n+i,n+j})$. The holomorphic mappings $\varphi : U \subset \subset \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ are now to be considered as the (local) sections of the fiber manifold $E$ satisfying the Cauchy-Riemann equations $R^1$

\[
y_{ij} - y_{n+i,n+j} = 0, \quad y_{i,n+j} + y_{n+i,j} = 0; \quad i, j = 1, \ldots, n. \]

Now, let $M^m \subset \mathbb{G}^n = \mathbb{R}^{2n}$ be an analytic submanifold, $p \in M^m$ its point. Consider the space

\[
\tau_p(M^m) = T_p(M^m) \cap I T_p(M^m). \]

This space is always of even dimension; let us restrict ourselves to submanifolds $M^m$ with dim $\tau_p(M^m) = 2q = \text{const}$. To the submanifold $M^m \subset \mathbb{G}^n$, we associate a $G$-structure $B_G(M^m)$ as follows. The tangent frame

\[
\{v_1, \ldots, v_q, v_{q+1}, \ldots, v_{2q}, v_{2q+1}, \ldots, v_m\}
\]

at the point $p \in M^m$ is situated in $B_G(M^m)$ if and only if $v_1, \ldots, v_{2q} \in \tau_p(M^m)$ and $v_{q+\alpha} = = Iv_{\alpha}$ for $\alpha = 1, \ldots, q$. We have the following
Theorem 3.1. Let $M^n, \bar{M}^n \subset \mathcal{C} = \mathbb{R}^{2n}$ be two analytic submanifolds, let
\[ \dim \tau_p(M^n) = \dim \tau_p(\bar{M}^n) = \text{const.} \quad \text{for} \quad p \in M^n, \quad \bar{p} \in \bar{M}^n. \]
Be given an analytic map $\varphi : M^n \to \bar{M}^n$ satisfying $\varphi_* B_G(M^n) = B_G(\bar{M}^n)$. Let $p_0 \in M^n$ be a fixed point.
Then there exists a neighbourhood $U \subset \mathbb{R}^{2n}$ of the point $p_0$ and a holomorphic mapping $\Phi : U \to \mathbb{R}^{2n}$ such that $\Phi|_{\tau_{p_0} U} = \varphi$.

This theorem follows directly from Theorem 2.3. It has been proved by B. ČENKL and myself for $m = 2n - 1$ and by J. VANŽURA for a general $m$; both papers are unpublished.

The proof of lemma 1.1 is easy.

Now, let $\varphi : \mathcal{C} \to \mathcal{C}$ be a (local) biholomorphic mapping given by $z' = z'(z)$, i.e.,
\[ x' + iy' = f(x, y) + ig(x, y). \]
The mapping $\varphi$ induces a mapping $\varphi^* : \mathbb{R}^2 \to \mathbb{R}^2$
given by $x' = f(x, y), \quad y' = g(x, y)$. We have
\[
A = \left| \frac{dz'}{dz} \right| = \left| \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) (f + ig) \right| = \left| \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right| = \left( \frac{\partial f}{\partial x} \right)^2 + \left( \frac{\partial f}{\partial y} \right)^2 \right|^{1/2},
\]
\[
D = \left| \begin{array}{cc}
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\
\frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}
\end{array} \right| = \left( \frac{\partial f}{\partial x} \right)^2 + \left( \frac{\partial g}{\partial y} \right)^2,
\]
i.e., $D = A^2$. Thus the mapping $\varphi$ satisfies $A = 1$ if and only if $D = 1$. A similar (and, of course, a more complicated) calculation shows that the same property takes place for biholomorphic mappings $\varphi : \mathbb{C}^2 \to \mathbb{C}^2$. In the associated space $\mathbb{R}^4$, we thus get a volume structure. On each hypersurface $M^3 \subset \mathbb{C}^2 = \mathbb{R}^4$ we naturally obtain, with respect to the pseudogroup $\Gamma_3$, a $G$-structure described in more detail in the next chapter. Its definition is as follows: a frame $\{v_1, v_2, v_3\}$ at the point $m \in M^3$ belongs to $B_G(M^3)$ if and only if $v_1 \in \tau_m, \quad v_2 = Iv_1, \quad v_3 \in T_m(M^3)$ and the volume $[v_1, v_2, v_3, Iv_3] = 1$.

4. THE INDUCED $G$-STRUCTURE

Be given a 3-dimensional manifold $M$ together with a $G$-structure $B_G$, $G$ being the group of matrices of the form
\[
\begin{pmatrix}
\alpha & -\beta & 0 \\
\beta & \alpha & 0 \\
\gamma & \delta & \varphi
\end{pmatrix}, \quad (\alpha^2 + \beta^2) \varphi^2 = 1.
\]
\[ \{v_1, v_2, v_3\} \quad \text{and} \quad \{w_1, w_2, w_3\} \]
being two frames of $B_G$ at $m \in M$, we have
\[
w_1 = \alpha v_1 - \beta v_2, \quad w_2 = \beta v_1 + \alpha v_2, \quad w_3 = \gamma v_1 + \delta v_2 + \varphi v_3;
\]
\[ 312 \]
the plane $\tau_m \subset T_m(M)$ spanned by the vectors $v_1, v_2$ is thus invariant as well the endomorphism $I_m^2 : \tau_m \to \tau_m$, $I_m^2 = -\text{id}$, determined by $I_mv_1 = v_2$, $I_mv_2 = -v_1$.

Consider a point $m_0 \in M$, its neighbourhood $U \subset M$, and two sections $\{v_1, v_2, v_3\}$ and $\{w_1, w_2, w_3\}$ of $B_G$ over $U$. We have (4.2), $\varphi, \ldots, \varphi$ being real-valued functions over $U$. Let us write

$$
\begin{align*}
&[v_1, v_2] = a_1v_1 + a_2v_2 + a_3v_3, \quad [w_1, w_2] = A_1w_1 + A_2w_2 + A_3w_3, \\
&[v_1, v_3] = b_1v_1 + b_2v_2 + b_3v_3, \quad [w_1, w_3] = B_1w_1 + B_2w_2 + B_3w_3, \\
&[v_2, v_3] = c_1v_1 + c_2v_2 + c_3v_3, \quad [w_2, w_3] = C_1w_1 + C_2w_2 + C_3w_3.
\end{align*}
$$

From the Jacobi identity

$$
[v_1, w_2] + [v_2, w_3] + [v_3, v_1] = 0,
$$
we get

$$
\begin{align*}
&v_1c_1 - v_2b_1 + v_3a_1 + a_1c_2 + b_1c_3 - b_2c_1 - a_2c_1 = 0, \\
v_1c_2 - v_2b_2 + v_3a_2 + a_2c_2 + b_2c_3 + b_1c_2 - a_1b_2 = 0, \\
v_1c_3 - v_2b_3 + v_3a_3 + a_3c_2 + b_3c_3 - a_1b_3 - a_2c_3 = 0
\end{align*}
$$
and analogous equations for $A_1, \ldots, C_3$. Let us study the relations existing between $a_1, \ldots, c_3$ and $A_1, \ldots, C_3$. We have

$$
\begin{align*}
&[w_1, w_2] = [\alpha v_1 - \beta v_2, \beta v_1 + \alpha v_2] = (\cdot) v_1 + (\cdot) v_2 + (x^2 + \beta^2) a_3v_3, \\
&[w_1, w_3] = [\alpha v_1 - \beta v_2, \gamma v_1 + \delta v_2 + \varphi v_3] = (\cdot) v_1 + (\cdot) v_2 + \\
&\quad + (\alpha, v_1, \varphi - \beta, v_2, \varphi + \alpha \delta a_3 + \alpha \varphi b_3 + \beta \gamma a_3 - \beta \varphi c_3) v_3, \\
&[w_2, w_3] = [\beta v_1 + \alpha v_2, \gamma v_1 + \delta v_2 + \varphi v_3] = (\cdot) v_1 + (\cdot) v_2 + \\
&\quad + (\beta, v_1, \varphi + \alpha, v_2, \varphi + \beta \delta a_3 + \beta \varphi b_3 - \gamma \alpha a_3 + \alpha \varphi c_3) v_3,
\end{align*}
$$

i.e.,

$$
\begin{align*}
&\varphi A_3 = (x^2 + \beta^2) a_3, \\
&\varphi B_3 = \alpha, v_1, \varphi - \beta, v_2, \varphi + \alpha \delta a_3 + \alpha \varphi b_3 + \beta \gamma a_3 - \beta \varphi c_3, \\
&\varphi C_3 = \beta, v_1, \varphi + \alpha, v_2, \varphi + \beta \delta a_3 + \beta \varphi b_3 - \gamma \alpha a_3 + \alpha \varphi c_3.
\end{align*}
$$

*Let us restrict ourselves to the case $a_3 \neq 0$, this being equivalent to the non-integrability of the field of the planes $\tau_m$. We get — from (4.5) — the possibility to choose the section $\{w_1, w_2, w_3\}$ in such a way that $A_3 = 1$, $B_3 = C_3 = 0$. Suppose that the section $\{v_1, v_2, v_3\}$ has been already chosen in such a manner that $a_3 = 1$, $b_3 = c_3 = 0$. Then the equation (4.51) reduces to $\varphi = x^2 + \beta^2$, and we get $\varphi = 1, x^2 + \beta^2 = 1$ from (4.1). The equations (4.52,3) reduce to $0 = \alpha \delta + \beta \gamma, 0 = \beta \delta - \gamma \alpha$, i.e., $\delta = \gamma = 0$. From this, we get*
Lemma 4.1. The considered G-structure $B_0$ may be reduced to the H-structure $B_H$ the sections $\{v_1, v_2, v_3\}$ of which satisfy

\begin{align}
[v_1, v_2] &= a_1 v_1 + a_2 v_2 + v_3, \\
[v_1, v_3] &= b_1 v_1 + b_2 v_2, \\
[v_2, v_3] &= c_1 v_1 + c_2 v_2,
\end{align}

$H$ being the group of matrices of the form

\begin{equation}
\begin{pmatrix}
\alpha & -\beta & 0 \\
\beta & \alpha & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad \alpha^2 + \beta^2 = 1.
\end{equation}

The equations (4.4) reduce to

\begin{align}
v_2c_1 - v_2b_1 + v_3a_1 + a_1c_2 - a_2c_1 &= 0, \\
v_1c_2 - v_2b_2 + v_3a_2 + a_2b_1 - a_1b_2 &= 0, \\
c_2 + b_1 &= 0.
\end{align}

Now, let $\{v_1, v_2, v_3\}$ and $\{w_1, w_2, w_3\}$ be two sections of the reduction $G_H$. Then

\begin{align}
[w_1, w_2] &= [\alpha v_1 - \beta v_2, \beta v_1 + \alpha v_2] = (\alpha v_1 \beta - \beta v_1 \alpha + a_1) v_1 + \\
&\quad + (-\beta v_2 \alpha + \alpha v_2 \beta + a_2) v_2 + v_3 = \\
&= (\alpha A_1 + \beta A_2) v_1 + (-\beta A_1 + \alpha A_2) v_2 + v_3, \\
[w_1, w_3] &= [\alpha v_1 - \beta v_2, v_3] = (-v_3 \alpha + \beta v_1 + \beta c_1) v_1 + \\
&\quad + (-v_3 \beta + \alpha b_1 + \alpha c_2) v_2 = \\
&= (\alpha B_1 + \beta B_2) v_1 + (-\beta B_1 + \alpha B_2) v_2, \\
[w_2, w_3] &= [\beta v_1 + \alpha v_2, v_3] = (-v_3 b + \beta b_1 + \alpha c_1) v_1 + \\
&\quad + (-v_3 \alpha + \alpha b_2 + \alpha c_2) v_2 = \\
&= (\alpha C_1 + \beta C_2) v_1 + (-\beta C_1 + \alpha C_2) v_2.
\end{align}

From the last two relations, we have

\begin{align}
\alpha(b_1 - c_2) - \beta(b_2 + c_1) &= \alpha(B_1 - C_2) + \beta(B_2 + C_1), \\
\beta(b_1 - c_2) + \alpha(b_2 + c_1) &= -\beta(B_1 - C_2) + \alpha(B_2 + C_1),
\end{align}

i.e.,

\begin{align}
B_1 - C_2 &= (\alpha^2 - \beta^2)(b_1 - c_2) - 2\alpha\beta(b_2 + c_1), \\
B_2 + C_1 &= 2\alpha\beta(b_1 - c_2) + (\alpha^2 - \beta^2)(b_2 + c_1).
\end{align}

Thus there exist sections $\{w_1, w_2, w_3\}$ such that $B_1 - C_2 = 0$. Suppose that the section $\{v_1, v_2, v_3\}$ has been already chosen in such a way that $b_1 - c_2 = 0$; from
(4.8), we get \( b_1 = c_2 = 0 \). Then

\[
\alpha \beta (b_2 + c_1) = 0, \quad B_2 + C_1 = (\alpha^2 - \beta^2)(b_2 + c_1).
\]

Suppose \( b_2 + c_1 \neq 0 \). Then \( \alpha \beta = 0 \), i.e., \( \alpha = 0, \beta = \varepsilon \) or \( \beta = 0, \alpha = \varepsilon \) resp.; \( \varepsilon = \pm 1 \).

**Lemma 4.2.** The considered G-structure \( B_0 \) may be reduced to a K-structure \( B_K \), the sections \( \{v_1, v_2, v_3\} \) of which fulfill

\[
\begin{aligned}
[v_1, v_2] &= a_1 v_1 + a_2 v_2 + v_3, \\
[v_1, v_3] &= b_2 v_2, \\
[v_2, v_3] &= c_1 v_1, \\
\end{aligned}
\]

\[
\begin{aligned}
v_1 c_1 + v_3 a_1 - a_2 c_1 &= 0, \\
v_2 b_2 + v_3 a_2 - a_1 b_2 &= 0.
\end{aligned}
\]

The relation \( b_2 + c_1 = 0 \) is invariant. If \( b_2 + c_1 \neq 0 \), \( K \) is the group of the matrices of the form

\[
\begin{pmatrix}
\varepsilon & 0 & 0 \\
0 & \varepsilon & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

with \( \varepsilon = \pm 1 \).

This lemma solves the equivalence problem for the G-structures with \( b_2 + c_1 \neq 0 \); in fact, to each such G-structure we have associated four \( \ell \)-structures, and the equivalence problem has been reduced to the equivalence problem of \( \ell \)-structures.

Now, let us consider the more complicated case \( b_2 + c_1 = 0 \). Let us write \( c_1 = q, b_2 = -q \); the equations (4.9) are now

\[
\begin{aligned}
[v_1, v_2] &= a_1 v_1 + a_2 v_2 + v_3, & [v_1, v_3] &= -q v_2, & [v_2, v_3] &= q v_1; \\
v_1 q + v_3 a_1 - a_2 q &= 0, & v_2 q + v_3 a_2 + a_1 q &= 0.
\end{aligned}
\]

Let \( \{w_1, w_2, w_3\} \) be another section of the reduced K-structure \( B_K \); suppose

\[
\begin{aligned}
w_1 &= \alpha v_1 - \beta v_2, & w_2 &= \beta v_1 + \alpha v_2, & w_3 &= v_3; & \alpha^2 + \beta^2 &= 1;
\end{aligned}
\]

and

\[
\begin{aligned}
[w_1, w_2] &= A_1 w_1 + A_2 w_2 + w_3, & [w_1, w_3] &= -Q w_2, & [w_2, w_3] &= Q w_1.
\end{aligned}
\]

We find

\[
\begin{aligned}
\beta a_1 &= v_1 \alpha + \alpha \beta A_1 + \beta^2 A_2, \\
\beta a_2 &= v_2 \alpha - \beta^2 A_1 + \alpha \beta A_2, \\
\beta q &= -v_3 \alpha + \beta Q,
\end{aligned}
\]

and it is easy to verify

315
Lemma 4.3. We have
\[ v_2a_1 - v_1a_2 + a_1^2 + a_2^2 - q = w_2A_1 - w_1A_2 + A_1^2 + A_2^2 - Q. \]

Let us write
\[ k = v_2a_1 - v_1a_2 + a_1^2 + a_2^2 - q, \quad K = w_2A_1 - w_1A_2 + A_1^2 + A_2^2 - Q. \]

Clearly,
\[ w_1K = \alpha \cdot v_1k - \beta \cdot v_2k, \quad w_2K = \beta \cdot v_1k + \alpha \cdot v_2k, \quad w_3K = v_3k. \]

Thus we have \((w_1K)^2 + (w_2K)^2 = (v_1k)^2 + (v_2k)^2\). If \(v_1k = 0, v_2k = 0\), we get \([v_1, v_2]k = 0\) and \(v_3k = 0\), i.e., \(k = \text{const}\). In the case \(k \neq \text{const}\), we are in the position to choose the section \(\{w_1, w_2, w_3\}\) in such a way that \(w_1K = 0\). Suppose, that the section \(\{v_1, v_2, v_3\}\) has been already chosen in this way. Then \(\beta = 0\) and \(\alpha = \varepsilon = \pm 1\), and we obtain

Lemma 4.4. Be given a G-structure \(B_G\), the reduction of which to the K-structure of Lemma 4.2 is such that \(b_2 + c_1 = 0\). Let \(k \neq \text{const}\). Then we are able to reduce our G-structure to the L-structure \(B_L\), the sections of which satisfy \(v_1k = 0\); \(L\) is the group of matrices
\[ (4.15) \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \varepsilon = \pm 1. \]

Thus we have reduced our study to the case \(k = \text{const}\). Be given a G-structure \(B_G\) by means of a section \(\{v_1, v_2, v_3\}\) satisfying (4.11) and \(v_1k = v_2k = v_3k = 0\). Consider the system of partial differential equations
\[ (4.16) \quad v_1\alpha = \beta a_1, \quad v_2\alpha = \beta a_2, \quad v_3\alpha = -\beta q - \beta k \]
for the unknown function \(\alpha, \beta\) being given by \(\alpha^2 + \beta^2 = 1\). It is easy to see that this system is completely integrable. Thus, there exists a section \(\{w_1, w_2, w_3\}\) of our G-structure such that \(A_1 = A_2 = 0, Q = -k\), and we have

Lemma 4.5. Be given a G-structure \(B_G\), and let its reduction to the K-structure of Lemma 4.2 be such that \(b_2 + c_1 = 0\) and \(k = \text{const}\). Then there are sections of \(B_K\) satisfying
\[ (4.17) \quad [v_1, v_2] = v_3, \quad [v_1, v_3] = kv_2, \quad [v_2, v_3] = -kv_1. \]

All other sections satisfying (4.17) are given by \(w_1 = \alpha v_1 - \beta v_2, w_2 = \beta v_1 + \alpha v_2, w_3 = v_3\), where \(\alpha^2 + \beta^2 = 1\) and \(\alpha = \text{const}\).

On the manifolds \(M\) and \(N\), be given G-structures \(B_G\) and \(B_G^1\) resp. of the type described in Lemma 4.5; suppose \(k = k'\). In a neighbourhood of a point \(m_0 \in M\),
let us choose a section of $B_Q$ satisfying (4.17), similarly, in a neighbourhood of a point $n_0 \in N$, let us choose a section $\{u_1, u_2, u_3\}$ satisfying analogous equations $[u_1, u_2] = u_3$, $[u_1, u_3] = ku_2$, $[u_2, u_3] = -ku_1$. Consider the manifold $M \times N$ and, in a suitable neighbourhood of the point $(m_0, n_0)$, the vector fields

$$V_1 = u_1^* + \alpha v_1^* - \beta v_2^*, \quad V_2 = u_2^* + \beta v_1^* + \alpha v_2^*, \quad V_3 = u_3^* + v_3^*;$$

here, $\alpha = $ const., $\alpha^2 + \beta^2 = 1$ and the vector fields $u_i^*, v_i^*$ are given by the conditions $(\pi_1) v_i^* = v_i$, $(\pi_2) v_i^* = 0$, $(\pi_1) u_i^* = u_i$, $(\pi_2) u_i^* = 0$, $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$ being the natural projections. It is easy to see that

$$[V_1, V_2] = V_3, \quad [V_1, V_3] = kV_2, \quad [V_2, V_3] = -kV_1.$$

Thus the distribution determined on a neighbourhood of $(m_0, n_0) \in M \times N$ by means of the vector fields $V_1, V_2, V_3$ is completely integrable, and it has an integral manifold going through the point $(m_0, n_0)$. This integral manifold is then a local diffeomorphism transforming $B_Q$ into $B_Q$.

Finally, let us investigate transitive $G$-structures. One type of these structures is given by Lemma 4.5. Consider the type given by Lemma 4.2 with $b_2 + c_1 \neq 0$. The functions $a_1, a_2, b_2, c_1$ being constant, we get $a_2c_1 = a_1b_2 = 0$ from (4.9). Thus we obtain

**Theorem 4.1.** Let $B_Q$ be a transitive $G$-structure on $M$. Then it is possible (in a suitable neighbourhood of each point $m_0 \in M$) to choose its section $\{v_1, v_2, v_3\}$ in such a way that

(4.18) $[v_1, v_2] = av_1 + v_3$, $[v_1, v_3] = 0$, $[v_2, v_3] = cv_1$; $a, c = \text{const.}, \quad c \neq 0$;

or

(4.19) $[v_1, v_2] = v_3$, $[v_1, v_3] = bv_2$, $[v_2, v_3] = cv_1$; $b, c = \text{const.}, \quad b \neq 0$, $c \neq 0$, $b + c \neq 0$;

or

(4.20) $[v_1, v_2] = v_3$, $[v_1, v_3] = kv_2$, $[v_2, v_3] = -kv_1$; $k = \text{const.}$;

respectively.

It is easy to verify that the transitive $G$-structures of all the types of Theorem 4.1 do exist. First of all, consider the $G$-structure of the type (4.18) on a manifold $M$. Let $m_0 \in M$, then there is a coordinate neighbourhood about $m_0$ with local coordinates $(x, y, z)$ such that

(4.21) $v_1 = \frac{\partial}{\partial x}$, $v_2 = (ax - cz) \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}$, $v_3 = \frac{\partial}{\partial z}$. 

317
To proceed further, a simple check shows us that the vector fields

\[
\begin{align*}
   u_1 &= \frac{1}{2}(1 + 2y - 3x^2) \frac{\partial}{\partial x} + \frac{1}{2}(2x + z - 3xy) \frac{\partial}{\partial y} + \frac{1}{2}(y - xz) \frac{\partial}{\partial z}, \\
   u_2 &= \frac{1}{2}(1 - 2y + 3x^2) \frac{\partial}{\partial x} + \frac{1}{2}(2x - z + 3xy) \frac{\partial}{\partial y} + \frac{1}{2}(y + xz) \frac{\partial}{\partial z}, \\
   u_3 &= x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y} + 3z \frac{\partial}{\partial z}
\end{align*}
\]

on \( \mathbb{R}^3 \) satisfy

\[
[u_1, u_2] = u_3, \quad [u_1, u_3] = u_2, \quad [u_2, u_3] = u_1.
\]

In a suitable neighbourhood of the point \((\frac{1}{4}\pi, 0, 0)\) \(\in \mathbb{R}^3\), consider the vectors fields

\[
\begin{align*}
   w_1 &= \sin (y + z) \frac{\partial}{\partial x} + \cos x \cos (y + z) \frac{\partial}{\partial y} - \sin x \cos (y + z) \frac{\partial}{\partial z}, \\
   w_2 &= \cos (y + z) \frac{\partial}{\partial x} - \cos x \sin (y + z) \frac{\partial}{\partial y} + \sin x \sin (y + z) \frac{\partial}{\partial z}, \\
   w_2 &= \frac{\partial}{\partial y} + \frac{\partial}{\partial z};
\end{align*}
\]

the direct check proves

\[
[w_1, w_2] = 2w_3, \quad [w_1, w_3] = -2w_2, \quad [w_2, w_3] = 2w_1.
\]

Now, the Lie algebra (4.20) with \( k = 0 \) is realized by (4.21) with \( a = c = 0 \). The Lie algebras (4.19) and (4.20) with \( k \neq 0 \) are of the form

\[
[v_1, v_2] = v_3, \quad [v_1, v_3] = Bv_2, \quad [v_2, v_3] = Cv_1; \quad BC \neq 0.
\]

The realizations of the Lie algebras (4.26) are as follows:

\[
\begin{align*}
   v_1 &= \sqrt{B} \cdot u_1, \quad v_2 = \sqrt{C} \cdot u_2, \quad v_3 = \sqrt{(BC)} \cdot u_3 \quad \text{for} \quad B > 0, \ C > 0, \\
   v_1 &= \sqrt{B} \cdot u_1, \quad v_2 = \sqrt{(-C)} \cdot u_3, \quad v_3 = \sqrt{(-BC)} \cdot u_2 \quad \text{for} \quad B > 0, \ C < 0, \\
   v_1 &= \frac{1}{2} \sqrt{(-B)} \cdot w_1, \quad v_2 = \frac{1}{2} \sqrt{C} \cdot w_2, \quad v_3 = \frac{1}{2} \sqrt{(-BC)} \cdot w_3 \quad \text{for} \quad B < 0, \ C > 0, \\
   v_1 &= \sqrt{(-B)} \cdot u_2, \quad v_2 = \sqrt{(-C)} \cdot u_1, \quad v_3 = -\sqrt{(BC)} \cdot u_3 \quad \text{for} \quad B < 0, \ C < 0.
\end{align*}
\]

Now, we are in the position to prove Theorem 1.1. The equivalence properties have been proved above. Now, let \( M \subset \mathbb{R}^2 \) be a 3-dimensional submanifold. It is clear that \( \dim G_3(M) \leq 4 \) and \( \dim G_3(M) = 4 \) if and only if the induced G-structure \( B_G \) over \( M \) is of the type (4.20). Thus, it is sufficient to prove
Theorem 4.2. The hypersurfaces $N^3_r, N^3_R, N^3_0 \subset \mathbb{R}^2$, i.e., the hypersurfaces $(z_1 = x_1 + iy_1, z_2 = x_2 + iy_2)$

(4.28)

$$
N^3_r : x_1^2 + y_1^2 + x_2^2 + y_2^2 = r^2 \quad (r > 0),
$$

$$
N^3_R : x_1 x_2 + y_1 y_2 = R \quad (R > 0),
$$

$$
N^3_0 : y_2^2 - 2y_1^2 = 0
$$

have the induced $G$-structure which is reducible to the $M$-structure $B_M$ (see Lemma 4.5) of the type (4.20) with

(4.29)

$$
k_r = -\frac{4}{\sqrt{(4r^2)}}, \quad k_R = \frac{1}{\sqrt{R^2}}, \quad k_0 = 0.
$$

Proof. First of all, consider the hypersurface $N^3_r$. On $\mathbb{R}^4$, consider the vector fields

(4.30)

$$
v_1 = \frac{1}{\sqrt{(2r)}} \left( y_2 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial y_2} \right),
$$

$$
v_2 = \frac{1}{\sqrt{(2r)}} \left( -x_2 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial y_1} + x_1 \frac{\partial}{\partial x_2} - y_1 \frac{\partial}{\partial y_2} \right),
$$

$$
v_3 = \frac{2}{\sqrt{(4r^2)}} \left( y_1 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial y_2} \right).
$$

These vector fields have the following properties: (i) they satisfy (4.20), $k$ being the $k_r$ given by (4.29); (ii) in the points of $N^3_r$, the considered vector fields are tangent to it; (iii) $Iv_1 = v_2$; (iv) $[v_1, v_2, v_3, Iv_3] = 1$ on $N^3_r$. For $N^3_R$ and $N^3_0$, we have similar results using the vector fields

(4.31)

$$
v_1 = \frac{1}{2 \sqrt{3}} \left( x_1 \frac{\partial}{\partial x_1} - y_1 \frac{\partial}{\partial y_1} - x_2 \frac{\partial}{\partial x_2} + y_2 \frac{\partial}{\partial y_2} \right),
$$

$$
v_2 = \frac{1}{2 \sqrt{3}} \left( y_1 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial y_1} - y_2 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial y_2} \right),
$$

$$
v_3 = \frac{1}{2 \sqrt{3} R^2} \left( - y_1 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial y_1} - y_2 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial y_2} \right)
$$

or

(4.32)

$$
v_1 = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial x_1} + 2y_1 \frac{\partial}{\partial x_2} \right), \quad v_2 = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial y_1} + 2y_1 \frac{\partial}{\partial y_2} \right), \quad v_3 = \frac{3}{2} \frac{\partial}{\partial x_2}
$$

respectively.
5. SIMPLY TRANSITIVE SUBMANIFOLDS

Consider the space \( \mathcal{C}^2 \) and the pseudogroup \( \Gamma_s \) of its local maps. The relation between the one-parametric local subgroups of the pseudogroup \( \Gamma \) and the holomorphic vector fields on \( \mathcal{C}^2 \) is well known. Let

\[
(5.1) \quad v = A(z_1, z_2) \frac{\partial}{\partial z_1} + B(z_1, z_2) \frac{\partial}{\partial z_2}
\]

be a holomorphic vector field on \( \mathcal{C}^2 \); the corresponding local group \( G_v \) consisting of the transformations

\[
(5.2) \quad \varphi_t : z_1 = f(z_1, z_2, t), \quad z_2 = g(z_1, z_2, t), \quad t \in (-\varepsilon, \varepsilon),
\]

is given by the differential equations

\[
(5.3) \quad \frac{\partial f(z_1, z_2, t)}{\partial t} = A(f(z_1, z_2, t), g(z_1, z_2, t)), \quad \frac{\partial g(z_1, z_2, t)}{\partial t} = B(f(z_1, z_2, t), g(z_1, z_2, t)),
\]

\[
f(z_1, z_2, 0) = z_1, \quad g(z_1, z_2, 0) = z_2.
\]

**Theorem 5.1.** Consider the vector field (5.1) on \( \mathcal{C}^2 \). Then \( G_v \subset \Gamma_s \) if and only if

\[
(5.4) \quad \text{Re} \left( \frac{\partial A(z_1, z_2)}{\partial z_1} + \frac{\partial B(z_1, z_2)}{\partial z_2} \right) = 0;
\]

here, \( \text{Re} z = \frac{1}{2}(z + \bar{z}) \).

**Proof.** Let us write

\[
D(z_1, z_2, t) = \frac{\partial f(z_1, z_2, t)}{\partial z_1} \frac{\partial g(z_1, z_2, t)}{\partial z_2} - \frac{\partial f(z_1, z_2, t)}{\partial z_2} \frac{\partial g(z_1, z_2, t)}{\partial z_1},
\]

we have \( D(z_1, z_2, 0) = 1 \). Then

\[
\frac{\partial D}{\partial t} = \frac{\partial^2 f}{\partial z_1 \partial t} \frac{\partial g}{\partial z_2} + \frac{\partial f}{\partial z_1} \frac{\partial^2 g}{\partial z_2 \partial t} - \frac{\partial^2 f}{\partial z_2 \partial t} \frac{\partial g}{\partial z_1} - \frac{\partial f}{\partial z_2} \frac{\partial^2 g}{\partial z_1 \partial t},
\]

from (5.3), we have

\[
\begin{align*}
\frac{\partial^2 f}{\partial z_1 \partial t} &= \frac{\partial A}{\partial z_1} \frac{\partial f}{\partial t} + \frac{\partial A}{\partial z_2} \frac{\partial g}{\partial t}, & \frac{\partial^2 f}{\partial z_2 \partial t} &= \frac{\partial A}{\partial z_1} \frac{\partial f}{\partial t} + \frac{\partial A}{\partial z_2} \frac{\partial g}{\partial t}, \\
\frac{\partial^2 g}{\partial z_1 \partial t} &= \frac{\partial B}{\partial z_1} \frac{\partial f}{\partial t} + \frac{\partial B}{\partial z_2} \frac{\partial g}{\partial t}, & \frac{\partial^2 g}{\partial z_2 \partial t} &= \frac{\partial B}{\partial z_1} \frac{\partial f}{\partial t} + \frac{\partial B}{\partial z_2} \frac{\partial g}{\partial t}.
\end{align*}
\]
Hence,

\[ \frac{\partial D}{\partial t} = \left( \frac{\partial A}{\partial z_1} + \frac{\partial B}{\partial z_2} \right) D \]

and

\[ \frac{\partial (D\bar{D})}{\partial t} = \left( \frac{\partial A}{\partial z_1} + \frac{\partial B}{\partial z_2} + \frac{\partial A}{\partial z_1} + \frac{\partial B}{\partial z_2} \right) D\bar{D}. \]

The theorem follows easily.

Now, let us study submanifolds \( M^3 \subset \mathbb{C}^2 \) for which the Lie algebra of the group \( G_2(M^3) \) is given by (1.18), i.e.,

\[ [v_1, v_2] = av_1 + v_3, \quad [v_1, v_3] = 0, \]
\[ [v_2, v_3] = cv_1; \quad c \neq 0. \]

**Lemma 5.1.** Let \( \mathcal{L}_s \) be the Lie algebra of vector fields (5.1) on \( \mathbb{C}^2 \) satisfying (5.4). Let \( L \subset \mathcal{L}_s \) be the subalgebra (5.5). Then \( a = 0 \) or \( v_3 = A(z_1, z_2) v_1 \) for some function \( A \).

**Proof.** It is always possible to choose the coordinates \( (z_1, z_2) \) of \( \mathbb{C}^2 \) in such a way that (at least locally)

\[ v_1 = \frac{\partial}{\partial z_1}. \]

Let

\[ v_3 = \alpha(z_1, z_2) \frac{\partial}{\partial z_1} + \beta(z_1, z_2) \frac{\partial}{\partial z_2}; \quad \frac{\partial \alpha}{\partial z_1} + \frac{\partial \beta}{\partial z_2} = xi, \quad x \in \mathbb{R}. \]

We have \( \partial \alpha/\partial z_1 = 0, \partial \beta/\partial z_1 = 0 \) from (5.5). Thus, the vector field \( v_3 \) may be written as

\[ v_3 = \alpha(z_2) \frac{\partial}{\partial z_1} + (xi z_2 + \lambda) \frac{\partial}{\partial z_2}; \quad x, \lambda \in \mathbb{R}. \]

Suppose \( x \neq 0 \). Let us choose a solution \( \mu(z_2) \) of the differential equation

\[ \frac{\partial \mu(z_2)}{\partial z_2} = -\frac{\alpha(z_2)}{xi z_2 + \lambda} \]

and the transformation of coordinates given by \( \zeta_1 = z_1 + \mu(z_2), \zeta_2 = z_2 - i\alpha^{1\cdot\lambda}. \)

Of course,

\[ \det \left( \frac{\partial (\zeta_1, \zeta_2)}{\partial (z_1, z_2)} \right) = 1, \]

321
and we have

\( v_1 = \frac{\partial}{\partial \xi_1}, \quad v_3 = \kappa i \xi_2 \frac{\partial}{\partial \xi_2} \)

in the new coordinates. Further, suppose \( \kappa = 0, \lambda \neq 0 \). In this case, let us consider the transformation of coordinates given by \( \xi_1 = z_1 + \mu(z_2), \xi_2 = z_2, \mu(z_2) \) being again a solution of (5.8). In the new coordinates,

\( v_1 = \frac{\partial}{\partial \xi_1}, \quad v_3 = \lambda \frac{\partial}{\partial \xi_2} \).

Thus, in suitable coordinates, the vector field \( v_3 \) may be written as

\( v_3 = \kappa iz_2 \frac{\partial}{\partial z_2} \) or \( v_3 = \lambda \frac{\partial}{\partial z_2} \) or \( v_3 = \kappa(z_2) \frac{\partial}{\partial z_1} \) respectively.

Suppose that \( v_3 \) is the vector field (5.11), let

\( v_2 = \varphi(z_1, z_2) \frac{\partial}{\partial z_1} + \sigma(z_1, z_2) \frac{\partial}{\partial z_2} ; \quad \frac{\partial \varphi}{\partial z_1} + \frac{\partial \sigma}{\partial z_2} = \kappa' i, \quad \kappa' \in \mathbb{R}. \)

From (5.5, 3), we obtain

\( \frac{\partial \varphi}{\partial z_1} = a, \quad \frac{\partial \sigma}{\partial z_1} = \kappa iz_2, \quad -\kappa iz_2 \frac{\partial \varphi}{\partial z_2} = c, \quad \sigma = y \frac{\partial \sigma}{\partial z_2}, \)

and we get

\( \frac{\partial \sigma}{\partial z_2} = \kappa' i - a, \quad \sigma = (\kappa' i - a) z_2, \quad \text{i.e.,} \quad \frac{\partial \sigma}{\partial z_1} = 0 \)

from (5.12) and (5.13). It follows from (5.13) that \( \kappa = 0 \), i.e., \( v_3 = 0 \), this being impossible. Further, suppose that \( v_3 \) is given by (5.11) and \( v_2 \) by (5.12). Then we obtain

\( \frac{\partial \varphi}{\partial z_1} = a, \quad \frac{\partial \sigma}{\partial z_1} = \lambda, \quad -\lambda \frac{\partial \varphi}{\partial z_2} = c, \quad \frac{\partial \sigma}{\partial z_2} = 0 \)

from (5.5, 3), and (5.14) yield \( a = \kappa' i \), i.e., \( a = \kappa' = 0 \) because of \( a \in \mathbb{R} \). Q.E.D.

**Theorem 5.2.** Consider the manifold \( N^3 \subset \mathbb{C}^2 \) given by

\( (z_1 - \bar{z}_1)^2 + c^3(z_2 - \bar{z}_2) + 4 = 0, \quad 0 \neq c \in \mathbb{R}. \)

Its group \( G_c(N^3) \) is

\( z'_1 = mz_1 - c^3 nz_2 + p, \quad z'_2 = nz_1 + mz_2 + q, \)

where \( m, n, p, q \in \mathbb{R}, m^2 + c^3 n^2 = 1. \)

322
Let $M^3$ be a manifold such that the Lie algebra of $G_s(M^3)$ is (5.5). Then $a = 0$, and the manifolds $M^3$ and $N^3$ are (locally) $\Gamma$-equivalent.

Proof. It is easy to see that (5.16) preserves $N^3$. Using the usual coordinates $z_i = x_i + iy_i$, $N^3$ as a submanifold of $\mathbb{R}^4$ is given by

$$y_1^2 + c^3 y_2^2 = 1.$$  

(5.17)

On $\mathbb{R}^4$, consider the vector fields

$$v_1 = c^2 y_2 \frac{\partial}{\partial x_1} - \frac{1}{c} y_1 \frac{\partial}{\partial x_2},$$

$$v_2 = c^2 y_2 \frac{\partial}{\partial y_1} - \frac{1}{c} y_1 \frac{\partial}{\partial y_2},$$

$$v_3 = cy_1 \frac{\partial}{\partial x_1} + cy_2 \frac{\partial}{\partial x_2}.$$  

(5.18)

We see easily that the vector fields (5.18) have the following properties: (i) $Iv_1 = v_2$; (ii) they satisfy (5.5) with $a = 0$; (iii) restricted to the points of $N^3$, they are tangent to it; (iv) we have $[v_1, v_2, v_3, Iv_3] = 1$ at the points of $N^3$. Thus $N^3$ is the model for manifolds $M^3$ of the type (5.5) with $a = 0$. Now suppose that $M^3$ admits the group $G_s(M^3)$, the Lie algebra of which is (5.5) with $a \neq 0$. The manifold $M^3$ may be constructed as follows. First of all, realize the Lie algebra $L(5.5)$ as a subalgebra $L \subset \mathcal{L}_x$. The vector fields $v_1, v_2, v_3$ being considered as vector fields on $\mathbb{R}^4$, they span an integrable 3-dimensional distribution $\mathcal{A}$. Now, $M^3$ is an integral manifold of $\mathcal{A}$. According to Lemma 5.1, we may choose the coordinates $(z_1, z_2)$ in $\mathbb{C}^2$ in such way that

$$v_1 = \frac{\partial}{\partial z_1}, \quad v_3 = \alpha(z_1, z_2) \frac{\partial}{\partial z_1} = (F + iG) \frac{\partial}{\partial z_1}.$$  

These vector fields regarded as vector fields on $\mathbb{R}^4$ are

$$v_1 = \frac{\partial}{\partial x_1}, \quad v_3 = F \frac{\partial}{\partial x_1} + G \frac{\partial}{\partial y_1}.$$  

(5.19)

Let

$$v_2 = A \frac{\partial}{\partial x_1} + B \frac{\partial}{\partial y_1} + C \frac{\partial}{\partial x_2} + D \frac{\partial}{\partial y_2}.$$  

(5.20)

The distribution $\mathcal{A}$ is determined by the vector fields (5.19) and (5.20). Let $M^3$ be its integral manifold. The plane $\tau_m$ being obviously spanned by the vectors $\partial/\partial x_1, \partial/\partial y_1$ at each point $m \in M^3$, the distribution $\{\tau_m\}$ is integrable. This is a contradiction as we have excluded such manifolds from our considerations. Q.E.D.
Let us now study manifolds $M^3$ such that the Lie algebra of $C_s(M^3)$ is of the type (1.19). First of all, let us prove several lemmas.

**Lemma 5.2. Consider the Lie algebra $L^+$**

\[(v_1, v_2) = v_2, \quad [v_1, v_3] = -v_3, \quad [v_2, v_3] = -2v_1\]

over $R$. The algebra $L^+$ is decomposed in a 1-parametric set of hyperboloids

\[H_k = \{xv_1 + yv_2 + zv_3; \quad x^2 - 4yz = k\}, \quad k \in R,\]

with the following property: Let $v \in H_k$, $v' \in H_{k'}$; $k = k'$ if and only there is an automorphism $\mathcal{A}: L^+ \to L^+$ satisfying $\mathcal{A}v = v'$. The vector

\[v_+ = \frac{1}{2}(v_2 - v_3)\sqrt{k} \quad \text{or} \quad v_- = \frac{1}{2}(v_2 + v_3)\sqrt{(-k)} \quad \text{or} \quad v_0 = v_2 \quad \text{respectively}\]

is situated in $H_k$ for $k > 0$ or $k < 0$ or $k = 0$ respectively.

**Proof.** $L^+$ has the following automorphism:

\[(A_1 v_1 = av_1 + bv_2 + cv_3, \quad A_1 v_2 = A\left(v_1 + \frac{b}{a - 1}v_2 + \frac{c}{a + 1}v_3\right), \quad A_1 v_3 = B\left(v_1 + \frac{b}{a + 1}v_2 + \frac{c}{a - 1}v_3\right), \quad a^2 \neq 1, \quad 4bc = a^2 - 1 = AB;\]

\[(A_2 v_1 = v_1 + av_2, \quad A_2 v_2 = bv_2, \quad A_2 v_3 = A(2av_1 + a^2v_2 + v_3), \quad Ab = 1;\]

\[(A_3 v_1 = v_1 + av_3, \quad A_3 v_2 = A(2av_1 + v_2 + a^2v_3), \quad A_3 v_3 = bv_3, \quad Ab = 1;\]

\[(A_4 v_1 = -v_1 + av_2, \quad A_4 v_2 = A(-2av_1 + a^2v_2 + v_3), \quad A_4 v_3 = bv_2, \quad Ab = 1;\]

\[(A_5 v_1 = -v_1 + av_3, \quad A_5 v_2 = bv_3, \quad A_5 v_3 = A(-2av_1 + v_2 + a^2v_3), \quad Ab = 1.\]

The vector $v = xv_1 + yv_2 + zv_3$ be called interior (or exterior resp.) if $x^2 - 4yz > 0$ (or $x^2 - 4yz > 0$ respectively); the set of interior (exterior) vectors be denoted by $H^+$ ($H^-$ respectively). (1) Consider the vector $v = \alpha v_1 + \beta v_2 + \gamma v_3 \notin H_0$. (1)
Let $\gamma \neq 0$. If $v \in H^+$, we have $4\beta\gamma - x^2 > 0$; choosing $\mathcal{A}_2$ (5.25) with

$$A = \frac{1}{2} \sqrt{4\beta\gamma - x^2}, \quad a = -\frac{x}{\sqrt{4\beta\gamma - x^2}}, \quad b = \frac{1}{A},$$

we obtain $\mathcal{A}_2 v = \frac{1}{2}(v_2 + v_3) \sqrt{(4\beta\gamma - x^2)}$. If $v \in H^-$, we have $x^2 - 4\beta\gamma > 0$; choosing $\mathcal{A}_2$ (5.25) with

$$A = -\frac{1}{2\gamma} \sqrt{(x^2 - 4\beta\gamma)}, \quad a = \frac{x}{\sqrt{(x^2 - 4\beta\gamma)}}, \quad b = \frac{1}{A},$$

we obtain $\mathcal{A}_2 v = \frac{1}{2}(v_2 - v_3) \sqrt{(x^2 - 4\beta\gamma)}$. (1) Let $\gamma = 0$, $\beta \neq 0$. Then $v \in H^-$; choosing $\mathcal{A}_3$ (5.26) with $A = \frac{1}{2}\| v \|$, $a = -\beta^{-1} \text{sgn } \alpha$, $b = A^{-1}$, we obtain $\mathcal{A}_3 v = \frac{1}{2}\| v \| \cdot (v_2 - v_3)$. (1) Let $\beta = 0$. Then $v \in H^-$. Choosing $\mathcal{A}_1$ (5.24) with $a = 0$, $b = \frac{1}{2} \text{sgn } \alpha$, $c = -\frac{1}{2} \text{sgn } \alpha$, $A = 1$, $B = -1$, we obtain $\mathcal{A}_1 v = \frac{1}{2}\| v \| \cdot (v_2 - v_3)$.

(2) Suppose $v \in H_0$, i.e., $4\beta\gamma - x^2 = 0$. (1) Let $\beta \neq 0$. Choosing $\mathcal{A}_3$ (5.26) with $A = \beta^{-1}$, $a = -\frac{1}{2}\alpha$, $b = \beta$, we have $\mathcal{A}_3 v = v_2$. (2) Let $\beta = 0$. Then $x = 0$; choosing $\mathcal{A}_4$ (5.27) with $A = \gamma$, $b = \gamma^{-1}$, $a = 0$, we obtain $\mathcal{A}_4 v = v_2$. Q.E.D.

**Lemma 5.3.** Let $(v, \tilde{v})$ be a couple of vectors of the Lie algebra $L^+$. For a suitable automorphism $\mathcal{A} : L^+ \to L^+$, the couple $w = \mathcal{A} v$, $\tilde{w} = \mathcal{A} \tilde{v}$ becomes one of the following couples:

(5.29) \[ w = k(v_2 - v_3), \quad \tilde{w} = l_1 v_2 + l_2 v_3, \quad k(l_1 + l_2) \neq 0; \]
\[ w = k(v_2 - v_3), \quad \tilde{w} = l_1 v_1 + l_2 (v_2 - v_3), \quad kl_1 \neq 0; \]
\[ w = k(v_2 - v_3), \quad \tilde{w} = l(v_1 \pm v_2), \quad kl \neq 0; \]
\[ w = k(v_2 - v_3), \quad \tilde{w} = 2v_1 + v_2 + v_3, \quad k \neq 0; \]
\[ w = k(v_2 - v_3), \quad \tilde{w} = l(v_2 - v_3), \quad k \neq 0; \]
\[ w = k(v_2 + v_3), \quad \tilde{w} = l_1 v_2 + l_2 v_3, \quad k \neq 0; \]
\[ w = k v_2, \quad \tilde{w} = l_1 v_2 + l_2 v_3, \quad kl_2 \neq 0; \]
\[ w = k v_2, \quad \tilde{w} = l v_1, \quad kl \neq 0; \]
\[ w = k v_2, \quad \tilde{w} = l v_2, \quad k \neq 0; \]
\[ w = 0, \quad \tilde{w} = k(v_2 - v_3); \]
\[ w = 0, \quad \tilde{w} = k(v_2 + v_3); \]
\[ w = 0, \quad \tilde{w} = k v_2. \]

**Proof.** The automorphisms $\mathcal{A} : L^+ \to L^+$ satisfying

(5.30) \[ \mathcal{A}(v_2 - v_3) = v_2 - v_3 \quad \text{or} \quad \mathcal{A}(v_2 + v_3) = v_2 + v_3 \quad \text{or} \quad \mathcal{A} v_2 = v_2 \]

respectively
are

\[ (5.31) \]
\[ \mathbf{Av}_1 = av_1 + ax_2 + ax_3, \]
\[ \mathbf{Av}_2 = 2ax_1 + \frac{1}{2}(a + 1)x_2 + \frac{1}{2}(a - 1)x_3, \]
\[ \mathbf{Av}_3 = 2ax_1 + \frac{1}{2}(a - 1)x_2 + \frac{1}{2}(a + 1)x_3, \quad a^2 - 1 = 4x^2, \]
or

\[ (5.32) \]
\[ \mathbf{Av}_1 = av_1 + ax_2 - ax_3, \]
\[ \mathbf{Av}_2 = -2ax_1 + \frac{1}{2}(a + 1)x_2 - \frac{1}{2}(a - 1)x_3, \]
\[ \mathbf{Av}_3 = 2ax_1 - \frac{1}{2}(a - 1)x_2 + \frac{1}{2}(a + 1)x_3, \quad 1 - a^2 = 4x^2, \]
or

\[ (5.33) \]
\[ \mathbf{Av}_1 = v_1 + av_2, \quad \mathbf{Av}_2 = v_2, \]
\[ \mathbf{Av}_3 = 2av_1 + a^2v_2 + v_3 \]
respectively. Be given a vector \( u = \varrho_1v_1 + \varrho_2v_2 + \varrho_3v_3 \in L^+ \). (i) There is an automorphism \( \mathcal{A} \) (5.31) such that

\[ (5.34) \]
\[ \mathcal{A}u = \sigma_1v_2 + \sigma_2v_3 \quad \text{for} \quad (\varrho_2 + \varrho_3)^2 - \varrho_1^2 > 0 \]
or

\[ (5.35) \]
\[ \mathcal{A}u = \sigma_3v_1 + \sigma_4(v_2 - v_3) \quad \text{for} \quad (\varrho_2 + \varrho_3)^2 - \varrho_1^2 < 0 \]
or

\[ (5.36) \]
\[ \mathcal{A}u = \sigma_5(v_1 \pm v_2) \quad \text{for} \quad (\varrho_2 + \varrho_3)^2 = \varrho_1^2, \quad \varrho_1(\varrho_2^2 - \varrho_3^2) \neq 0 \]
or

\[ (5.37) \]
\[ \mathcal{A}u = 2v_1 + v_2 + v_3 \quad \text{for} \quad \varrho_2 = \varrho_3, \quad \varrho_1^2 = 4\varrho_2^2, \quad \varrho_2 \neq 0 \]
respectively. Indeed, use the automorphism \( \mathcal{A} \) (5.31) with

\[ a = -(\varrho_2 + \varrho_3) \{ (\varrho_2 + \varrho_3)^2 - \varrho_1^2 \}^{-1/2}, \quad \alpha = \frac{1}{2} \varrho_1 \{ (\varrho_2 + \varrho_3)^2 - \varrho_1^2 \}^{-1/2} \]
or

\[ a = \varrho_1 \{ \varrho_1^2 - (\varrho_2 + \varrho_3)^2 \}^{-1/2}, \quad \alpha = -\frac{1}{2} (\varrho_2 + \varrho_3) \{ \varrho_1^2 - (\varrho_2 + \varrho_3)^2 \}^{-1/2} \]
or

\[ a = (\varrho_2^2 + \varrho_3^2) (\varrho_2^2 - \varrho_3^2)^{-1}, \quad \alpha = -\varrho_1^{-1} \varrho_2 \varrho_3 (\varrho_2 - \varrho_3)^{-1} \]
or

\[ a = \frac{1}{2} \varrho_2^{-1} (\varrho_2^2 + 1), \quad \alpha = \frac{1}{2} \varrho_2^{-1} (\varrho_2^2 - 1) \]
respectively. (ii) There exists an automorphism \( \mathcal{A} \) (5.32) such that

\[ (5.38) \]
\[ \mathcal{A}u = \sigma_6v_2 + \sigma_7v_3 \quad \text{for} \quad \varrho_1^2 + (\varrho_2 - \varrho_3)^2 \neq 0; \]

it is sufficient to take

\[ a = (\varrho_2 - \varrho_3) \{ \varrho_1^2 + (\varrho_2 - \varrho_3)^2 \}^{-1/2}, \quad \alpha = \frac{1}{2} \varrho_1 \{ \varrho_1^2 + (\varrho_2 - \varrho_3)^2 \}^{-1/2}. \]
For \( q_1 = 0, q_3 = q_2 \), we have \( \mathcal{A}u = u \) for each automorphism \( \mathcal{A} \) (5.32). (iii) There exists an automorphism \( \mathcal{A} \) (5.33) such that

\[
(5.39) \quad \mathcal{A}u = \sigma_8 v_2 + \sigma_9 v_3 \quad \text{for} \quad q_3 \neq 0
\]
or

\[
(5.40) \quad \mathcal{A}u = \sigma_{10} v_1 \quad \text{for} \quad q_3 = 0, \quad q_1 \neq 0
\]
respectively; it is sufficient to take

\[
a = -\frac{1}{2} q_1 q_3^{-1} \quad \text{or} \quad a = -q_2 q_1^{-1}
\]
respectively.

The lemma follows from what has been and from the preceding lemma.

**Theorem 5.3.** Let \( \mathcal{L} \) be the Lie algebra of vector fields (5.1) satisfying (5.4) on \( \mathcal{C}^2 \). Then, in a neighbourhood of a fixed point \( m \in \mathcal{C}^2 \), we may choose holomorphic coordinates \( (z_1, z_2) \) such that \( m = (0, 1) \),

\[
(5.41) \quad v_1 = z_1 \frac{\partial}{\partial z_1} - z_2 \frac{\partial}{\partial z_2}, \quad v_3 = \frac{\partial}{\partial z_1}
\]
and \( v_2 \) is one of the following vector fields:

\[
(5.42) \quad v_2 = \left(z_1^2 + \frac{p + iq}{z_2^2}\right) \frac{\partial}{\partial z_1} - 2z_1z_2 \frac{\partial}{\partial z_2},
\]

\[
v_2 = \left(z_1^2 + \frac{p + iq}{z_2^2}\right) \frac{\partial}{\partial z_1} + \left(-2z_1z_2 - \frac{qr \pm qt}{p}\right) \frac{\partial}{\partial z_2}, \quad p, q \neq 0,
\]

\[
v_2 = \left(z_1^2 + \frac{s - i}{(s + i)z_2^2}\right) \frac{\partial}{\partial z_1} + \left(-2z_1z_2 + \frac{2}{s^2 + 1} + i \frac{2s}{s^2 + 1}\right) \frac{\partial}{\partial z_2}, \quad s \neq 0,
\]

\[
v_2 = \left(z_1^2 + \frac{1}{z_2^2}\right) \frac{\partial}{\partial z_1} + \left(-2z_1z_2 + p + iq\right) \frac{\partial}{\partial z_2},
\]

\[
v_2 = \left(z_1^2 - \frac{1}{z_2^2}\right) \frac{\partial}{\partial z_1} - 2z_1z_2 \frac{\partial}{\partial z_2},
\]

\[
v_2 = z_1^2 \frac{\partial}{\partial z_1} + \left(-2z_1z_2 + ip\right) \frac{\partial}{\partial z_2}; \quad p, q, r, s \in \mathcal{R}.
\]

**Proof.** In a neighbourhood of the point \( m \in \mathcal{C}^2 \), let us choose coordinates in such a way that (5.41). Now, let

\[
(5.43) \quad v_1 = a \frac{\partial}{\partial z_1} + b \frac{\partial}{\partial z_2}, \quad \frac{\partial a}{\partial z_1} + \frac{\partial b}{\partial z_2} = xi, \quad x \in \mathcal{R}.
\]
From (5.212), we get the existence of functions \( \varphi(z_2), \psi(z_2) \) such that \( a = z_1 + \varphi(z_2), \ b = \psi(z_2) \); the condition (5.432) assures the existence of a constant \( \lambda \in \mathcal{C} \) such that

\[
(5.44) \quad v_1 = (z_1 + \varphi(z_2)) \frac{\partial}{\partial z_1} + ((\kappa i - 1) z_2 + \lambda \frac{\partial}{\partial z_2}.
\]

Let us consider a change of coordinates

\[
\zeta_1 = z_1 + \Phi(z_2), \quad \zeta_2 = z_2 + \frac{\lambda}{\kappa i - 1},
\]

\( \Phi(z_2) \) being a solution of the differential equation

\[
\{(\kappa i - 1) z_2 + \lambda\} \Phi'(z_2) = \Phi(z_2) \varphi(z_2).
\]

We get by a direct calculation

\[
v_3 = \frac{\partial}{\partial \zeta_1}, \quad v_4 = \zeta_1 \frac{\partial}{\partial \zeta_1} + (\kappa i - 1) \zeta_2 \frac{\partial}{\partial \zeta_2};
\]

writing \( \zeta_i = z_i \), we get

\[
(5.45) \quad v_1 = z_1 \frac{\partial}{\partial z_1} + (\kappa i - 1) z_2 \frac{\partial}{\partial z_2}.
\]

Let

\[
(5.46) \quad v_2 = \alpha \frac{\partial}{\partial z_1} + \beta \frac{\partial}{\partial z_2}, \quad \frac{\partial \alpha}{\partial z_1} + \frac{\partial \beta}{\partial z_2} = \kappa' i, \quad \kappa' \in \mathcal{R}.
\]

From (5.213), we get

\[
\frac{\partial \alpha}{\partial z_1} = 2z_1, \quad \frac{\partial \beta}{\partial z_1} = 2(\kappa i - 1) z_2,
\]

i.e., the existence of functions \( f(z_2), g(z_2) \) such that

\[
\alpha = z_1^2 + f(z_2), \quad \beta = 2(\kappa i - 1) z_1 z_2 + g(z_2).
\]

From (5.462), we get \( 2\kappa i z_1 + g'(z_2) = \kappa i, \) i.e.,

\[
(5.47) \quad \kappa = 0.
\]

Thus, \( g(z_2) = \kappa' i z_2 + d \); from (5.211),

\[
(5.48) \quad z_2 f'(z_2) = -2 f(z_2), \quad \kappa' = 0,
\]

i.e., we have (5.40) and

\[
(5.49) \quad v_2 = \left(z_1^2 + \frac{c}{z_2}\right) \frac{\partial}{\partial z_1} + \left(-2z_1 z_2 + d\right) \frac{\partial}{\partial z_2}; \quad c, \ d \in \mathcal{C}.
\]

328
Thus we have proved that we may choose (at least locally) coordinates in such a way that the vector fields $v_1, v_2, v_3$ satisfying (5.21) are given by (5.40) and (5.49). But there is the possibility to choose another basis of $L^+$ satisfying (5.21). We have

$$v_2 = \left(2z_1 - \frac{d}{z_2}\right)v_1 + \left(-\frac{z_2^2}{z_1} + \frac{c}{z_2} + d\frac{z_1}{z_2}\right)v_3;$$

in the point $z_1 = 0, z_2 = 1$,

$$v_2 + dv_1 - cv_3 = 0.$$

The choice of the new basis in $L^+$ is now to be done in such a way that (5.50) has the canonical form $w + i\tilde{w} = 0$, $w$ and $\tilde{w}$ being given by (5.29). Q.E.D.

**Lemma 5.4.** Let $L^+$ be the Lie algebra (5.21). Let $L \supset L^+$ be a Lie algebra with $\dim L = 4$. Then there is a vector $v_4 \in L - L^+$ and numbers $a_2, a_3, b_2 \in \mathbb{R}$ such that $L$ is given by (5.21) and

$$[v_1, v_4] = a_2v_2 + a_3v_3,$$

$$[v_2, v_4] = 2a_3v_1 + b_2v_2,$$

$$[v_3, v_4] = 2a_2v_1 - b_2v_3.$$

**Proof.** We may write

$$[v_1, v_4] = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4,$$

$$[v_2, v_4] = b_1v_1 + b_2v_2 + b_3v_3 + b_4v_4,$$

$$[v_3, v_4] = c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4.$$

Consider the Jacobi identities

$$[v_1, [v_2, v_4]] + [v_2, [v_4, v_1]] + [v_4, [v_1, v_2]] = 0,$$

$$[v_1, [v_3, v_4]] + [v_3, [v_4, v_1]] + [v_4, [v_1, v_3]] = 0,$$

$$[v_2, [v_3, v_4]] + [v_3, [v_4, v_2]] + [v_4, [v_2, v_3]] = 0.$$

From (5.52) and (5.53), we get (5.51). Q.E.D.

**Lemma 5.5.** Consider the Lie algebra $L^+ \subset L_3$ given by the vector fields (5.41) and (5.49). The Lie algebra $L$ satisfying $L^+ \subset L \subset L_3$ and $\dim L = 4$ exists if and only if $c = d = 0$.

**Proof.** Let $L$ exist, let

$$v_4 = A \frac{\partial}{\partial z_1} + B \frac{\partial}{\partial z_2}, \quad \frac{\partial A}{\partial z_1} + \frac{\partial B}{\partial z_2} = \kappa i, \quad \kappa \in \mathbb{R}.$$

329
From (5.51),
\[
\frac{\partial A}{\partial z_1} = 2a_2z_1 - b_2, \quad \frac{\partial B}{\partial z_1} = -2a_2z_2,
\]
from (5.54),
\[
\frac{\partial B}{\partial z_2} = \kappa i - 2a_2z_1 + b_2.
\]
Thus, there is a function \( \phi(z_2) \) and a constant \( \lambda \in \mathbb{C} \) such that
\[
A = a_2z_1^2 - b_2z_1 + \phi(z_2), \quad B = -2a_2z_1z_2 + \kappa iz_2 + b_2z_2 + \lambda.
\]
From (5.51), we get
\[
\frac{\partial A}{\partial z_2} = a_2z_1^2 + a_2c - b_2z_1 + a_3.
\]
\[
\frac{\partial B}{\partial z_2} = a_2d,
\]
i.e., there exists a number \( \mu \in \mathbb{C} \) such that
\[
A = a_2z_1^2 - b_2z_1 + a_2c - b_2z_1 + a_3 + \mu \frac{1}{z_2},
\]
\[
B = -2a_2z_1z_2 + \kappa iz_2 + b_2z_2 + a_2d.
\]
Finally, we obtain from (5.51)
\[
\left( z_1^2 + \frac{c}{z_2^2} \right) (2a_2z_1 - b_2) + \left( -2z_1z_2 + d \right) \frac{\partial A}{\partial z_2} = 2A z_1 + 2Bc \frac{1}{z_2} =
\]
\[
= 2a_3z_1 + b_2 \left( z_1^2 + \frac{c}{z_2^2} \right),
\]
\[
-2a_2z_1^2z_2 - 2a_2c \frac{1}{z_2} + d \frac{\partial B}{\partial z_2} + 2z_1 \left( B - z_2 \frac{\partial B}{\partial z_2} \right) + 2A z_2 =
\]
\[
= -2a_3z_2 - 2b_2z_1z_2 + b_2d,
\]
i.e.,
(5.55) \[ a_3 = \mu = \kappa d = c \kappa = 0. \]
Suppose \( a_3 = \mu = \kappa = 0 \). Then \( v_4 = a_2v_2 - b_2v_1 \) and \( \dim L = 3 \). Hence \( a_3 = \mu = c = d = 0 \) and
(5.56) \[ v_4 = a_2v_2 - b_2v_1 + j\kappa z_2 \frac{\partial}{\partial z_2}. \]
Of course, we have to suppose \( \kappa \neq 0 \). Q.E.D.
Theorem 5.4. Let $M^3 \subset \mathcal{C}^2$ be a hypersurface, let the Lie algebra of the group $G_3(M^3)$ be of the type

$$[u_1, u_2] = u_3, \quad [u_1, u_3] = bu_2, \quad [u_2, u_3] = cu_1,$$

$c \neq 0, \quad b + c \neq 0, \quad b > 0$ or $b < 0, \quad c < 0$.

Then $M^3$ is an orbit of the group $G$ generated by the fields (5.41), (5.42); the field

$$u_2 = z_1^2 \frac{\partial}{\partial z_1} - 2z_1z_2 \frac{\partial}{\partial z_2}$$

is to be excluded.

Proof. If $b > 0$, let us choose a new basis

$$v_1 = \frac{1}{\sqrt{b}} u_1, \quad v_2 = \frac{1}{c} u_2 + \frac{1}{\sqrt{bc}} u_3, \quad v_3 = u_2 - \frac{1}{\sqrt{b}} u_3,$$

if $b < 0, c < 0$, consider the basis

$$v_1 = -\frac{1}{\sqrt{(bc)}} u_3, \quad v_2 = -\frac{1}{b} u_1 - \frac{1}{\sqrt{(bc)}} u_2, \quad v_3 = -\frac{1}{\sqrt{c}} u_2.$$

Then (5.21) is satisfied and the theorem follows from Theorem 5.3 and Lemma 5.5.

6. TRANSITIVE SUBMANIFOLDS $M^4 \subset \mathcal{C}^3$

In $\mathcal{C}^3$, consider the complex coordinates $z_i = x_i + iy_i; i = 1, 2, 3$. The space $\mathcal{C}^3$ be identified with $\mathbb{R}^6$ in the usual way. Thus, $(\partial/\partial x_i, \partial/\partial y_i)$ is the basis of $\mathbb{R}^6$ and the known endomorphism $I : \mathbb{R}^6 \rightarrow \mathbb{R}^6, I^2 = -\text{id.}$, is given by

$$I \frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i}, \quad I \frac{\partial}{\partial y_i} = -\frac{\partial}{\partial x_i}; \quad i = 1, 2, 3.$$

In $\mathcal{C}^3 = \mathbb{R}^6$, consider a real submanifold $M^4$. Write $\tau_m = T_m(M^4) \cap IT_m(M^4)$, $T_m(M^4)$ being the tangent space of $M^4$ at $m \in M^4$. Suppose dim $\tau_m = 2$ for each $m \in M^4$. In the principal fiber bundle $R(M^4)$ of the frames over $M^4$, let us choose (locally) a section $\sigma = (v_1, v_2, v_3, v_4)$ in such a way that $v_1(m) \in \tau_m$ and $Iv_1 = v_2$.

The section $\tilde{\sigma} = (w_1, w_2, w_3, w_4)$ having the same property, we have

$$(6.1) \quad v_1 = \alpha w_1 - \beta w_2, \quad v_2 = \beta w_1 + \alpha w_2,$$

$$v_3 = \gamma w_1 + \delta w_2 + \phi w_3 + \psi w_4,$$

$$v_4 = A w_1 + B w_2 + C w_3 + D w_4$$

with $(\alpha^2 + \beta^2)(\phi D - \psi C) \neq 0$. 

331
We may write

\[(6.2) \quad [v_1, v_2] = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4,\]
\[(v_1, v_3) = b_1v_1 + b_2v_2 + b_3v_3 + b_4v_4,\]
\[(v_1, v_4) = c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4,\]
\[(v_2, v_3) = d_1v_1 + d_2v_2 + d_3v_3 + d_4v_4,\]
\[(v_2, v_4) = e_1v_1 + e_2v_2 + e_3v_3 + e_4v_4,\]
\[(v_3, v_4) = f_1v_1 + f_2v_2 + f_3v_3 + f_4v_4,\]

the functions \(a_1, \ldots, f_4\) satisfying the Jacobi identities

\[(6.3) \quad [v_1, [v_2, v_3]] + [v_2, [v_3, v_1]] + [v_3, [v_1, v_2]] = 0,\]
\[[v_1, [v_2, v_4]] + [v_2, [v_4, v_1]] + [v_4, [v_1, v_2]] = 0,\]
\[[v_1, [v_3, v_4]] + [v_3, [v_4, v_1]] + [v_4, [v_1, v_3]] = 0,\]
\[[v_2, [v_3, v_4]] + [v_3, [v_4, v_2]] + [v_4, [v_2, v_3]] = 0.\]

Let \(p \in M^4\) be a fixed point, \(v_0 = A_0 v_1(p) + B_0 v_2(p) \in \tau_p\) a given vector. Let us choose a vector field \(v = Av_1 + Bv_2\) such that \(v(m) \in \tau_m\) for each \(m \in M^4\), and suppose \(v(p) = v_0\). Then \(Iv = -Bv_1 + Av_2\), and we have

\[[v, Iv] = [Av_1 + Bv_2, -Bv_1 + Av_2] = (\cdot) v_1 + (\cdot) v_2 + (A^2 + B^2)(a_3v_3 + a_4v_4).\]

If \(L^{(1)}_p \neq 0\), we do not have \(a_3 = a_4 = 0\). Thus, we are in the position to choose \(\sigma\) in such a way that \(a_4 = 0, a_3 \neq 0\). The space \(\sigma_p\) (for which definition see §1) is spanned by the vectors \(v_1(p), v_2(p), v_3(p)\). Further, we have

\[[v, [v, Iv]] = (\cdot) v_1 + (\cdot) v_2 + (\cdot) v_3 + (A_4 + B_4)(A^2 + B^2)a_3v_4.\]

If \(L^{(2)}_p \neq 0\), we do not have \(b_4 = d_4 = 0\). For the vector \(v' = d_4v_1 - b_4v_2, L^{(2)}_p(v') = 0\). Let us choose the section \(\sigma\) in such a way that \(L^{(2)}_p(v_2) = 0\), i.e., \(d_4 = 0, b_4 \neq 0\).

Thus, we consider over \(M^4\) only sections \(\sigma = (v_1, v_2, v_3, v_4)\) satisfying (6.2) with \(a_4 = d_4 = 0, a_3 \neq 0, b_4 \neq 0\). For any other section \(\tilde{\sigma} = (w_1, w_2, w_3, w_4)\) with the same property, we have (6.1) with \(\beta = 0, \psi = 0\) and, of course,

\[(6.3) \quad [w_1, w_2] = \tilde{a}_1w_1 + \tilde{a}_2w_2 + \tilde{a}_3w_3, \ldots,\]
\[[w_3, w_4] = \tilde{f}_1w_1 + \tilde{f}_2w_2 + \tilde{f}_3w_3 + \tilde{f}_4w_4.\]

Now,

\[[v_1, v_2] = [\alpha w_1, \alpha w_2] = (\cdot) w_1 + (\cdot) w_2 + \alpha^2 \tilde{a}_3w_3 = (\cdot) w_1 + (\cdot) w_2 + \varphi a_3w_3,\]

332
\[ [v_1, v_3] = [\omega w_1, \gamma w_1 + \delta w_2 + \phi w_3] = (\cdot) w_1 + (\cdot) w_2 + (\cdot) w_3 + \alpha \phi \bar{b}_4 w_4 =
\]
\[ = (\cdot) w_1 + (\cdot) w_2 + (\cdot) w_3 + D b_4 w_4 ,
\]
and we have \( \alpha^2 \bar{a}_3 = \phi a_3, \alpha \phi \bar{b}_4 = Db_4 \). The section \( \sigma \) may be chosen in such a way that \( a_3 = 1, b_4 = 1; \bar{a}_3 = \bar{b}_4 = 1 \) implies \( \phi = \alpha^2, D = \alpha^3 \).

Over \( M^4 \), we thus consider sections \( \sigma \) satisfying

\[ (6.4) \quad [v_1, v_2] = a_1 v_1 + a_2 v_2 + v_3 ,
\]
\[ [v_1, v_3] = b_1 v_1 + b_2 v_2 + b_3 v_3 + v_4 ,
\]
\[ [v_1, v_4] = c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 ,
\]
\[ [v_2, v_3] = d_1 v_1 + d_2 v_2 + d_3 v_3 ,
\]
\[ [v_2, v_4] = e_1 v_1 + e_2 v_2 + e_3 v_3 + e_4 v_4 ,
\]
\[ [v_3, v_4] = f_1 v_1 + f_2 v_2 + f_3 v_3 + f_4 v_4 .
\]

For another section \( \tilde{\sigma} \) with the same properties, we have

\[ (6.5) \quad v_1 = \omega w_1 ,
\]
\[ v_2 = \omega w_2 ,
\]
\[ v_3 = \gamma w_1 + \delta w_2 + \alpha^2 w_3 ,
\]
\[ v_4 = A w_1 + B w_2 + C w_3 + \alpha^3 w_4 , \quad \alpha \neq 0 .
\]

Now,

\[ [v_1, v_2] = [\omega w_1, \omega w_2] = \alpha(\omega \bar{a}_1 - w_2 \omega) w_1 + \alpha(\omega \bar{a}_2 + w_1 \omega) w_2 + \alpha^2 w_3 =
\]
\[ = (\omega a_1 + \gamma) w_1 + (\omega a_2 + \delta) w_2 + \alpha^2 w_3 ,
\]
\[ [v_1, v_3] = [\omega w_1, \gamma w_1 + \delta w_2 + \alpha^2 w_3] =
\]
\[ = (\cdot) w_1 + (\cdot) w_2 + (2 \omega w_1 \omega + \delta + \alpha^2 \bar{b}_3) w_3 + \alpha^2 w_4 =
\]
\[ = (\cdot) w_1 + (\cdot) w_2 + (\alpha^2 b_3 + C) w_3 + \alpha^3 w_4 ,
\]
\[ [v_2, v_3] = [\omega w_2, \gamma w_1 + \delta w_2 + \alpha^2 w_3] =
\]
\[ = (\cdot) w_1 + (\cdot) w_2 + (2 \omega w_2 \omega - \gamma + \alpha^2 \bar{a}_3) w_3 =
\]
\[ = (\cdot) w_1 + (\cdot) w_2 + \alpha^2 d_3 w_3 ,
\]
\[ [v_1, v_4] = [\omega w_1, A w_1 + B w_2 + \alpha^3 w_4] =
\]
\[ = (\cdot) w_1 + (\cdot) w_2 + (\cdot) w_3 + \alpha(3 \alpha^2 w_1 \omega + C + \alpha^3 \bar{c}_4) w_4 =
\]
\[ = (\cdot) w_1 + (\cdot) w_2 + (\cdot) w_3 + \alpha^3 c_4 w_4 ,
\]

333
and we obtain
\[ \alpha w_1 \alpha + \alpha^2 \bar{a}_2 = \alpha a_2 + \delta, \]
\[ -\alpha w_2 \alpha + \alpha^2 \bar{a}_1 = \alpha a_1 + \gamma, \]
\[ 2\alpha^2 w_1 \alpha + \alpha \delta + \alpha^3 \bar{b}_3 = \alpha^2 b_3 + C, \]
\[ 2\alpha w_2 \alpha - \gamma + \alpha^2 \bar{d}_3 = \alpha d_3, \]
\[ 3\alpha^2 w_1 \alpha + C + \alpha^3 \bar{c}_3 = \alpha^2 c_4 \]
and
\[ 3\alpha \delta + \alpha^3 (\bar{b}_3 - 2\bar{a}_2) = \alpha^2 (b_3 - 2a_2) + C, \]
\[ -3\gamma + \alpha^2 (\bar{d}_3 + 2\bar{a}_1) = \alpha (d_3 + 2a_1), \]
\[ C + \alpha^3 (\bar{c}_4 - 3\bar{a}_2) = \alpha^2 (c_4 - 3a_2) - 3\alpha \delta. \]

Thus we are in the position to choose a section \( \sigma \) in such a way that
\[ (6.6) \quad b_3 = 2a_2, \quad d_3 = -2a_1, \quad c_4 = 3a_2. \]
\( \sigma \) being another section satisfying (6.6), we have (6.5) with
\[ (6.7) \quad \gamma = \delta = C = 0. \]

Now,
\[ [v_1, v_3] = [\alpha w_1, \alpha^2 w_3] = \alpha^2 (-w_3 \alpha + \alpha \bar{b}_1) w_1 + \alpha^3 \bar{b}_2 w_2 + \]
\[ + 2\alpha^2 (w_1 \alpha + \alpha \bar{a}_2) w_3 + \alpha^3 w_4 = \]
\[ = (b_1 + A) w_1 + (\alpha b_2 + B) w_2 + 2\alpha^2 a_2 w_3 + \alpha^3 w_4, \]
\[ [v_2, v_3] = [\alpha w_2, \alpha^2 w_3] = \alpha^3 \bar{d}_1 w_1 + \alpha^2 (-w_3 \alpha + \alpha \bar{d}_2) w_2 + 2\alpha^2 (w_2 \alpha - \alpha \bar{a}_1) w_3 = \]
\[ = \alpha d_1 w_1 + \alpha d_2 w_2 - 2\alpha^2 a_1 w_3; \]
from these relations, we get
\[ \alpha^2 \bar{b}_2 = \alpha b_2 + B \]
and
\[ \alpha^3 (\bar{b}_1 - \bar{d}_2) = \alpha (b_1 - d_2) + A. \]

The section \( \sigma \) may be chosen in such a way that
\[ (6.8) \quad b_2 = 0, \quad d_2 = b_1; \]
\( \sigma \) being another section with the same properties, we have (6.5) with (6.7) and
\[ (6.9) \quad A = B = 0. \]

334
Lemma 6.1. Let $M^4 \subset \mathbb{R}^3$ be a submanifold with $\dim \tau_p = 2$, $L_p^{(1)} \neq 0$, $L_p^{(2)} \neq 0$ for each $p \in M^4$. Then there is a section $\sigma = (v_1, v_2, v_3, v_4)$ of $R(M^4)$ such that $v_2 = Iv_1$ and

\begin{align}
[v_1, v_2] &= a_1 v_1 + a_2 v_2 + v_3, \\
[v_1, v_3] &= b_1 v_1 + 2a_2 v_3 + v_4, \\
[v_1, v_4] &= c_1 v_1 + c_2 v_2 + c_3 v_3 + 3a_2 v_4, \\
[v_2, v_3] &= d_1 v_1 + b_1 v_2 - 2a_1 v_3, \\
[v_2, v_4] &= e_1 v_1 + e_2 v_2 + e_3 v_3 + e_4 v_4, \\
[v_3, v_4] &= f_1 v_1 + f_2 v_2 + f_3 v_3 + f_4 v_4.
\end{align}

$\sigma = (w_1, w_2, w_3, w_4)$ being another section with the same properties, we have

\begin{align}
v_1 &= \alpha w_1, \quad v_2 = \alpha w_2, \quad v_3 = \alpha^2 w_3, \quad v_4 = \alpha^3 w_4; \quad \alpha \neq 0.
\end{align}

Now,

\begin{align}
[v_1, v_2] &= [\alpha w_1, \alpha w_2] = \alpha(\alpha a_1 - w_2 \alpha) w_1 + \alpha(\alpha a_2 + w_1 \alpha) w_2 + \alpha^2 w_3 = \\
&= \alpha a_1 w_1 + \alpha a_2 w_2 + \alpha^2 w_3,
\end{align}

i.e.,

\begin{align}
[w_2 \alpha + \alpha a_1 = a_1, \quad w_1 \alpha + \alpha a_2 = a_2; \\
[v_1, v_3] &= [\alpha w_1, \alpha^2 w_3] = \alpha^2(-w_3 \alpha + \alpha a_1) w_1 + 2\alpha^2(\alpha a_2 + w_1 \alpha) w_3 + \alpha^3 w_4 = \\
&= \alpha b_1 w_1 + 2\alpha^2 a_2 w_3 + \alpha^3 w_4,
\end{align}

i.e.,

\begin{align}
-\alpha w_3 \alpha + \alpha^2 b_1 = b_1;
\end{align}

\begin{align}
[v_1, v_4] &= [\alpha w_1, \alpha^3 w_4] = \alpha^2(-w_4 \alpha + \alpha a_1) w_1 + \alpha^4 c_2 w_2 + \alpha^4 c_3 w_3 + \\
&+ 3\alpha^3(w_4 \alpha + \alpha a_2) w_4 = \alpha c_1 w_1 + \alpha c_2 w_2 + \alpha^2 c_3 w_3 + 3\alpha^3 a_2 w_4,
\end{align}

i.e.,

\begin{align}
\alpha^3 c_2 &= c_2, \quad \alpha^2 c_3 = c_3,
\end{align}

\begin{align}
-\alpha^2 w_4 \alpha + \alpha^3 c_1 = c_1;
\end{align}

\begin{align}
[v_2, v_3] &= [\alpha w_2, \alpha^2 w_3] = \alpha^3 d_1 w_1 + \alpha^2(-w_3 \alpha + \alpha a_1) w_2 + 2\alpha^2(w_2 \alpha - \alpha a_1) w_3 = \\
&= \alpha d_1 w_1 + \alpha b_1 w_2 - 2\alpha^2 a_1 w_3,
\end{align}

i.e.,

\begin{align}
\alpha^2 d_1 &= d_1;
\end{align}

\begin{align}
[v_2, v_4] &= [\alpha w_2, \alpha^3 w_4] = \alpha^4 c_1 w_1 + \alpha^3(-w_4 \alpha + \alpha a_2) w_2 + \alpha^4 c_3 w_3 + \\
&+ \alpha^3(3w_2 \alpha + \alpha a_4) w_4 = \alpha c_1 w_1 + \alpha c_2 w_2 + \alpha^2 c_3 w_3 + \alpha^3 e_4 w_4,
\end{align}
i.e.,

\[(6.17)\]
\[
\alpha^3 \varepsilon_1 = e_1, \quad \alpha^2 \varepsilon_3 = e_3,
\]
\[
\alpha^3 (\varepsilon_1 - \varepsilon_2) = c_1 - e_2, \quad \alpha(3\bar{a}_1 + \bar{e}_4) = 3a_1 + e_4
\]

by means of (6.15) and (6.12);

\[
[v_3, v_4] = [\alpha^3 w_3, \alpha^3 w_4] = \alpha^4 f_1 w_1 + \alpha^4 f_2 w_2 + \alpha^4 (-2w_4 x + \alpha f_3) w_3 +
\]
\[
+ \alpha^4 (3w_3 x + \alpha f_3) w_4 = \alpha f_1 w_1 + \alpha f_2 w_2 + \alpha^2 f_3 w_3 + \alpha^3 f_4 w_4,
\]

i.e.,

\[(6.18)\]
\[
\alpha^4 f_1 = f_1, \quad \alpha^4 f_2 = f_2,
\]
\[
\alpha^3 (f_3 - 2\varepsilon_1) = f_3 - 2c_1, \quad \alpha^2 (f_4 + 3\bar{b}_1) = f_4 + 3b_1
\]

by means of (6.15) and (6.13).

In what follows, let us restrict ourselves to manifolds \(M^4\) with \(\dim G(M^4) > 4\). Consider the equation (6.14). If \(c_2 \neq 0\), we are able to specialize the section \(\sigma\) in such a way that \(c_2 = 1\). We see at once that there is exactly one section \(\sigma\) satisfying (6.10) with \(c_2 = 1\); in fact, we have \(\alpha = 1\) from \(c_2 = \bar{c}_2 = 1\). This section is clearly preserved by \(G(M^4)\), hence \(\dim G(M^4) \leq 4\). Thus \(\dim G(M^4) > 4\) implies \(c_2 = 0\).

From similar reasons, we get

**Lemma 6.2.** Let \(M^4 \subset \mathcal{G}^3\) be a submanifold with \(\dim \tau_p = 2, \ I_p^{(1)} \neq 0, \ I_p^{(2)} \neq 0\) for each \(p \in M^4\), suppose \(\dim G(M^4) > 4\). Then there exists a section \(\sigma = (v_1, v_2, v_3, v_4)\) such that \(v_2 = \lambda v_1\) and

\[(6.19)\]
\[
[v_1, v_2] = a_1 v_1 + a_2 v_2 + v_3,
\]
\[
[v_1, v_3] = b_1 v_1 + 2a_2 v_3 + v_4,
\]
\[
[v_1, v_4] = c_1 v_1 + 3a_2 v_4,
\]
\[
[v_2, v_3] = b_1 v_2 - 2a_1 v_3,
\]
\[
[v_2, v_4] = c_1 v_2 - 3a_1 v_4,
\]
\[
[v_3, v_4] = 2c_1 v_3 - 3b_1 v_4.
\]

For another section \(\tilde{\sigma}\) with the same properties, we have

\[(6.20)\]
\[
v_1 = \alpha w_1, \quad v_2 = \alpha w_2, \quad v_3 = \alpha^2 w_3, \quad v_4 = \alpha^3 w_4;
\]

336
further,

\begin{align*}
(6.21) \quad w_1 x + x \tilde{a}_2 &= a_2, \quad x w_3 x - x^2 \tilde{b}_1 = -b_1, \\
& \quad w_2 x - x \tilde{a}_1 = -a_1, \quad x^2 w_4 x - x^3 \tilde{c}_1 = -c_1.
\end{align*}

The functions \( a_1, a_2, b_1, c_1 \) satisfy

\begin{align*}
(6.22) \quad v_3 a_1 - v_2 b_1 - a_1 b_1 &= 0, \quad v_3 a_2 + v_1 b_1 - a_2 b_1 - c_1 = 0, \\
& \quad v_1 a_1 + v_2 a_2 - b_1 = 0, \quad v_4 a_1 - v_2 c_1 - 2 a_1 c_1 = 0, \\
& \quad v_4 a_2 + v_1 c_1 - 2 a_2 c_1 = 0, \quad v_4 b_1 - v_3 c_1 - b_1 c_1 = 0.
\end{align*}

The equations (6.22) follow directly from (6.3). Consider now the system of partial differential equations

\begin{align*}
(6.23) \quad v_1 x &= x a_2, \quad v_2 x = -x a_1, \quad v_3 x = -x b_1, \quad v_4 x = -x c_1
\end{align*}

for \( \alpha \). Its integrability conditions are exactly (6.22), i.e., the system is completely integrable and its solution \( x \) is determined by the value \( x(m_0) \) at a fixed point \( m_0 \in M^4 \). From this and from (6.21), we obtain

**Theorem 6.1.** Let \( M^4 \subset \mathcal{G}^3 \) be a submanifold with \( \dim \tau_p = 2, \ L_p \not\equiv 0, \ L_p^{(2)} \not\equiv 0 \) for each \( p \in M^4 \), \( \dim G(M^4) > 4 \). Then there is a section \( \sigma = (v_1, v_2, v_3, v_4) \) such that \( v_2 = Iv_1 \) and

\begin{align*}
(6.24) \quad [v_1, v_2] &= v_3, \quad [v_1, v_3] = v_4, \\
& \quad [v_1, v_4] = [v_2, v_3] = [v_2, v_4] = [v_3, v_4] = 0.
\end{align*}

Any other section \( \tilde{\sigma} \) of the same type is given by

\begin{align*}
(6.25) \quad v_1 &= \alpha w_1, \quad v_2 = \alpha w_2, \quad v_3 = \alpha^2 w_3, \quad v_4 = \alpha^3 w_4; \quad 0 \neq \alpha = \text{const}.
\end{align*}

**Hence,** \( \dim G(M^4) = 5. \)

It is obvious that two manifolds of the type described in this Theorem are (locally) \( \Gamma \)-equivalent.

Consider the manifold \( N^4 \subset \mathcal{G}^3 \) given by

\begin{align*}
(6.26) \quad \bar{z}_2 - z_2 &= i(z_1 - z_1)^2, \quad \bar{z}_3 - z_3 = (z_1 - z_1)^3.
\end{align*}

Considering it as a submanifold of \( \mathcal{R}^6 \), its equations are

\begin{align*}
(6.27) \quad y_2 &= 2 y_1^2, \quad y_3 = -4 y_1^3;
\end{align*}
here, \( z_i = x_i + iy_i \). On \( \mathbb{R}^6 \), consider the vector fields

\begin{align*}
(6.28) \\
v_1 &= \frac{\partial}{\partial y_1} + 4y_1 \frac{\partial}{\partial y_2} - 6y_2 \frac{\partial}{\partial y_3} , \\
v_2 &= -\frac{\partial}{\partial x_1} - 4y_1 \frac{\partial}{\partial x_2} + 6y_2 \frac{\partial}{\partial x_3} , \\
v_3 &= -4 \frac{\partial}{\partial x_2} + 24y_1 \frac{\partial}{\partial x_3} , \\
v_4 &= 24 \frac{\partial}{\partial x_3} .
\end{align*}

It is easy to see that \( v_2 = Iv_1 \) and we have (6.23). Further, the vectors (6.27) are tangent to \( N^4 \) at its points. Thus we have proved Theorem 1.2.

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