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# SURFACES IN HERMITIAN 3-SPACES

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The differential geometry of submanifolds in hermitian spaces is not yet well known, the only exception being the theory of the curves due to O. BORŮVKA. In what follows, I propose to study a surface in $H^3$. To each point of it, I associate a geometrically significant frame, this giving me two "principal curvatures" denoted by $A$ and $C$. The theorems say that these curvatures should be "general" functions.

In the hermitian space $H^3$ be given an analytic surface $M = M(u, v)$, $u = u^1 + iu^2$, $v = v^1 + iv^2$ being local complex parameters. We have $\mathrm{d}M = \partial M/\partial u \,.\, \mathrm{d}u + \partial M/\partial v \,.\, \mathrm{d}v$, where $\partial/\partial u = \frac{1}{2}(\partial/\partial u^1 - i\partial/\partial u^2)$, $\mathrm{d}u = \mathrm{d}u^1 + i\,\mathrm{d}u^2$, etc. The tangent plane at each point of the surface $M$ is spanned by the vectors $\partial M/\partial u$, $\partial M/\partial v$. At each point of $M$, let us choose an orthonormal frame $v_1, v_2, v_3$ such that $v_1$ and $v_2$ are situated in the tangent plane. The field of these frames is supposed to be differentiable, but not generally holomorphic. Then we have

$$\frac{\partial M}{\partial u} = \alpha_1 v_1 + \alpha_2 v_2\,, \quad \frac{\partial M}{\partial v} = \beta_1 v_1 + \beta_2 v_2\,; \quad D \equiv \alpha_1\beta_2 - \alpha_2\beta_1 \neq 0\,;$$

and $\mathrm{d}M = \tau^1 v_1 + \tau^2 v_2$, where $\tau^1 = \alpha_1\,\mathrm{d}u + \beta_1\,\mathrm{d}v$, $\tau^2 = \alpha_2\,\mathrm{d}u + \beta_2\,\mathrm{d}v$. Now,

$$\tau^1 \wedge \tau^2 \wedge \bar{\tau}^1 \wedge \bar{\tau}^2 = D\bar{D}\,\mathrm{d}u \wedge \mathrm{d}v \wedge \mathrm{d}\bar{u} \wedge \mathrm{d}\bar{v} =$$
$$= 4D\bar{D}\,\mathrm{d}u^1 \wedge \mathrm{d}u^2 \wedge \mathrm{d}v^1 \wedge \mathrm{d}v^2 \neq 0\,.$$

Let us write

$$\text{(1)} \qquad \mathrm{d}M = \tau^1 v_1 + \tau^2 v_2\,, \qquad \mathrm{d}v_2 = \tau_2^1 v_1 + \tau_2^2 v_2 + \tau_2^3 v_3\,,$$
$$\mathrm{d}v_1 = \tau_1^1 v_1 + \tau_1^2 v_2 + \tau_1^3 v_3\,, \quad \mathrm{d}v_3 = \tau_3^1 v_1 + \tau_3^2 v_2 + \tau_3^3 v_3\,.$$

From the relations $(v_i, v_j) = \delta_{ij}$ ($\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$), we get

$$\text{(2)} \qquad \tau_i^j + \bar{\tau}_j^i = 0\,; \quad i, j = 1, 2, 3\,;$$

the exterior differentiation of (1) yields

$$(3)\qquad d\tau^1 = \tau^1 \wedge \tau_1^1 + \tau^2 \wedge \tau_2^1\,,\quad d\tau^2 = \tau^1 \wedge \tau_1^2 + \tau^2 \wedge \tau_2^2\,,$$
$$0 = \tau^1 \wedge \tau_1^3 + \tau^2 \wedge \tau_2^3\,;\quad d\tau_i^j = \tau_i^k \wedge \tau_k^j\,;\quad i, j, k = 1, 2, 3\,.$$

From $(3_3)$, we have the existence of complex-valued functions $A$, $B$, $C$ on $M$ such that

$$(4)\qquad \tau_1^3 = A\tau^1 + B\tau^2\,,\quad \tau_2^3 = B\tau^1 + C\tau^2\,.$$

At each point of $M$, let us choose another frame $w_1$, $w_2$, $w_3$ with the above described properties. Then

$$(5)\qquad dM = \omega^1 w_1 + \omega^2 w_2\,,\qquad dw_2 = \omega_2^1 w_1 + \omega_2^2 w_2 + \omega_2^3 w_3\,,$$
$$dw_1 = \omega_1^1 w_1 + \omega_1^2 w_2 + \omega_1^3 w_3\,,\quad dw_3 = \omega_3^1 w_1 + \omega_3^2 w_2 + \omega_3^3 w_3\,,$$

$$(6)\qquad \omega_1^3 = A^*\omega^1 + B^*\omega^2\,,\quad \omega_2^3 = B^*\omega^1 + C^*\omega^2\,;$$

we are interested in the relation between A, B, C and A*, B*, C*. Let

$$(7)\qquad v_1 = a_1 w_1 + a_2 w_2\,,\quad v_2 = b_1 w_1 + b_2 w_2\,,\quad v_3 = c w_3\,;$$
$$a_1\bar{a}_1 + a_2\bar{a}_2 = 1\,,\quad b_1\bar{b}_1 + b_2\bar{b}_2 = 1\,,\quad a_1\bar{b}_1 + a_2\bar{b}_2 = 0\,,\quad c\bar{c} = 1\,.$$

Then $\omega^1 = a_1\tau^1 + b_1\tau^2$, $\omega^2 = a_2\tau^1 + b_2\tau^2$, $\tau^1 = \bar{a}_1\omega^1 + \bar{a}_2\omega^2$, $\tau^2 = \bar{b}_1\omega^1 + \bar{b}_2\omega^2$. From $(1_{2,3})$, we get

$$a_1\, dw_1 + a_2\, dw_2 = (\cdot)\, w_1 + (\cdot)\, w_2 + (A\tau^1 + B\tau^2)\, c w_3\,,$$
$$b_1\, dw_1 + b_2\, dw_2 = (\cdot)\, w_1 + (\cdot)\, w_2 + (B\tau^1 + C\tau^2)\, c w_3\,,$$

i.e.,

$$a_1(A^*\omega^1 + B^*\omega^2) + a_2(B^*\omega^1 + C^*\omega^2) = A(\bar{a}_1\omega^1 + \bar{a}_2\omega^2)\, c + B(\bar{b}_1\omega^1 + \bar{b}_2\omega^2)\, c\,,$$
$$b_1(A^*\omega^1 + B^*\omega^2) + b_2(B^*\omega^1 + C^*\omega^2) = B(\bar{a}_1\omega^1 + \bar{a}_2\omega^2)\, c + C(\bar{b}_1\omega^1 + \bar{b}_2\omega^2)\, c$$

and

$$a_1A^* + a_2B^* = c\bar{a}_1A + c\bar{b}_1B\,,\quad a_1B^* + a_2C^* = c\bar{a}_2A + c\bar{b}_2B\,,$$
$$b_1A^* + b_2B^* = c\bar{a}_1B + c\bar{b}_1C\,,\quad b_1B^* + b_2C^* = c\bar{a}_2B + c\bar{b}_2C\,.$$

Finally,

$$(8)\qquad A^* = c(a_1b_2 - a_2b_1)^{-1}\,.\,(\bar{a}_1b_2A + \bar{b}_1b_2B - \bar{a}_1a_2B - a_2\bar{b}_1C)\,,$$
$$B^* = c(a_1b_2 - a_2b_1)^{-1}\,.\,(\bar{a}_2b_2A - a_2\bar{a}_2B + b_2\bar{b}_2B - a_2\bar{b}_2C)\,,$$
$$C^* = c(a_1b_2 - a_2b_1)^{-1}\,.\,(-\bar{a}_2b_1A + a_1\bar{a}_2B - b_1\bar{b}_2B + a_1\bar{b}_2C)\,.$$

Let $B \neq 0$. Consider the equation (for $\varrho \in \mathbf{C}$)

$$(9) \qquad \varrho\bar{\varrho} + \alpha\varrho - \beta\bar{\varrho} - 1 = 0\,, \quad \text{where} \quad \alpha = AB^{-1}\,, \quad \beta = -CB^{-1}\,.$$

Then $\varrho\bar{\varrho} - \bar{\beta}\varrho + \bar{\alpha}\bar{\varrho} - 1 = 0$; from these equations we get an equivalent system

$$(10) \qquad (\alpha + \bar{\beta})\,\varrho = (\bar{\alpha} + \beta)\,\bar{\varrho}\,, \quad \varrho\bar{\varrho} + \tfrac{1}{2}(\alpha - \bar{\beta})\,\varrho + \tfrac{1}{2}(\bar{\alpha} - \beta)\,\bar{\varrho} - 1 = 0\,.$$

In the plane of complex numbers, $(10_1)$ is the equation of a line through origin. $(10_2)$ may be written as

$$[\varrho + \tfrac{1}{2}(\bar{\alpha} - \beta)]\,.\,[\bar{\varrho} + \tfrac{1}{2}(\alpha - \bar{\beta})] = 1 + \tfrac{1}{4}(\bar{\alpha} - \beta)\,(\alpha - \bar{\beta})\,,$$

and it is the equation of a circle. The origin being its inner point, there exist solutions of (9). Let $\varrho_0$ be a solution of the equation (9). Further, choose $\beta$ in such a way that $\beta\bar{\beta}(1 + \varrho_0\bar{\varrho}_0) = 1$. Let the transformation (7) of the frames be given by

$$(11) \qquad v_1 = -\bar{\varrho}_0\beta w_1 - \beta w_2\,, \quad v_2 = \beta w_1 - \varrho_0\beta w_2\,, \quad v_3 = cw_3\,;$$

we see easily that the relations $(7_{4,5,6})$ are satisfied. We get

$$B^* = \frac{cB}{\beta^2(1 + \varrho_0\bar{\varrho}_0)^2}\left(\varrho_0\bar{\varrho}_0 + \frac{A}{B}\,\varrho_0 + \frac{C}{B}\,\bar{\varrho}_0 - 1\right) = 0\,,$$

and we have proved the existence of fields of frames $w_1, w_2, w_3$ such that $B^* = 0$. From now on, consider only the fields with this property. Thus $B = 0$, $B^* = 0$, and the equations (8) reduce to

$$(12) \qquad \begin{aligned} A^* &= c(a_1b_2 - a_2b_1)^{-1}\,(\bar{a}_1b_2A - a_2\bar{b}_1C)\,,\\ C^* &= c(a_1b_2 - a_2b_1)^{-1}\,(-\bar{a}_2b_1A + a_1\bar{b}_2C)\,, \end{aligned}$$

$$(13) \qquad 0 = \bar{a}_2b_2A - a_2\bar{b}_2C\,.$$

From (13) we get $a_2\bar{b}_2\bar{A} = \bar{a}_2b_2\bar{C}$, i.e., $a_2\bar{a}_2b_2\bar{b}_2(A\bar{A} - C\bar{C}) = 0$.

*Suppose* $A\bar{A} \neq C\bar{C}$. Then either $a_2 = 0$ or $b_2 = 0$. The admissible changes of the frames are

$$(14) \qquad \begin{aligned} &v_1 = a_1w_1\,, \quad v_2 = b_2w_2\,, \quad v_3 = cw_3\,; \quad a_1\bar{a}_1 = b_2\bar{b}_2 = c\bar{c} = 1\,;\\ &v_1 = a_2w_2\,, \quad v_2 = b_1w_1\,, \quad v_3 = cw_3\,; \quad a_2\bar{a}_2 = b_1\bar{b}_1 = c\bar{c} = 1\,, \end{aligned}$$

and we have

$$(15) \qquad A^* = c\,\frac{\bar{a}_1}{a_1}\,A\,, \quad C^* = c\,\frac{\bar{b}_2}{b_2}\,C \quad \text{or} \quad A^* = c\,\frac{\bar{b}_1}{b_1}\,C\,, \quad C^* = c\,\frac{\bar{a}_2}{a_2}\,A$$

and

$$(16) \qquad A^*\bar{A}^* = A\bar{A}\,, \quad C^*\bar{C}^* = C\bar{C} \quad \text{or} \quad A^*\bar{A}^* = C\bar{C}\,, \quad C^*\bar{C}^* = A\bar{A}\,.$$

The restriction to non-developpable surfaces leads to $AC \neq 0$, the asymptotic curves being given by the equation $\tau^1\tau_1^3 + \tau^2\tau_2^3 = A(\tau^1)^2 + C(\tau^2)^2 = 0$. We get-from (15)-the possibility to choose such fields of frames $v_1, v_2, v_3$ that $A > 0$, $C > 0$; let us call such fields canonical.

Now, *suppose* $A\bar{A} = C\bar{C}$. From (12), we get $A^*\bar{A}^* = A\bar{A}$, and we are able to choose a field of frames in such a way that $A = C > 0$.

Thus we are able — in any case — to choose the frames in such a way that

$$(17) \qquad \tau_1^3 = A\tau^1\,, \quad \tau_2^3 = C\tau^2\,; \quad A > 0\,, \quad C > 0\,.$$

The exterior differentation yields

$$(18) \qquad \begin{aligned} &\tau^1 \wedge \{dA + A(\tau_3^3 - 2\tau_1^1)\} + \tau^2 \wedge (A\bar{\tau}_1^2 - C\tau_1^2) = 0\,, \\ &\tau^1 \wedge (A\bar{\tau}_1^2 - C\tau_1^2) + \tau^2 \wedge \{dC + C(\tau_3^3 - 2\tau_2^2)\} = 0\,, \end{aligned}$$

and there exist complex-valued functions $K, L, M, N$ such that

$$(19) \qquad \begin{aligned} dA + A(\tau_3^3 - 2\tau_1^1) &= K\tau^1 + L\tau^2\,, \\ A\bar{\tau}_1^2 - C\tau_1^2 &= L\tau^1 + M\tau^2\,, \\ dC + C(\tau_3^3 - 2\tau_2^2) &= M\tau^1 + N\tau^2\,. \end{aligned}$$

From this, we get

$$(20) \qquad \begin{aligned} dA + A(2\tau_1^1 - \tau_3^3) &= \bar{K}\bar{\tau}^1 + \bar{L}\bar{\tau}^2\,, \\ A\tau_1^2 - C\bar{\tau}_1^2 &= \bar{L}\bar{\tau}^1 + \bar{M}\bar{\tau}^2\,, \\ dC + C(2\tau_2^2 - \tau_3^3) &= \bar{M}\bar{\tau}^1 + \bar{N}\bar{\tau}^2\,. \end{aligned}$$

Thus

$$(21) \quad 2dA = K\tau^1 + L\tau^2 + \bar{K}\bar{\tau}^1 + \bar{L}\bar{\tau}^2\,, \quad 2dC = M\tau^1 + N\tau^2 + \bar{M}\bar{\tau}^1 + \bar{N}\bar{\tau}^2\,.$$

Suppose $A = C \neq 0$. From (21), we get $K = M$, $L = N$, and $(19_{1,3})$ yields $\tau_2^2 - \tau_1^1 = 0$. From $(19_2)$ and $(20_2)$, we have $L = K = 0$ and $\bar{\tau}_1^2 = \tau_1^2$. From $\tau_2^2 - \tau_1^1 = 0$, we have $A^2(\tau^1 \wedge \bar{\tau}^1 - \tau^2 \wedge \bar{\tau}^2) = 0$, which is in contradiction to $\tau^1 \wedge \tau^2 \wedge \bar{\tau}^1 \wedge \bar{\tau}^2 \neq 0$. This proves

**Theorem 1.** *In $H^3$ there are no surfaces with $A = C$.*

Now, suppose $A \neq C$, $A =$ const., $C =$ const. From $(19_{1,3})$ and $(20_{1,3})$, $K = L = 0$, $M = N = 0$; from $(19_2)$ and $(20_2)$, $\tau_1^2 = 0$. The exterior differentation of this equation yields $AC\tau^1 \wedge \bar{\tau}^2 = 0$, and we have

**Theorem 2.** *In $H^3$ there are no surfaces with $A =$ const., $C =$ const.*

We get from (14) that at each point of our surface we have two invariant tangent directions which are analoguous to the principal directions of a surface in the

Euclidean space. If $v_1, v_2$ are tangent vectors of curves of the considered surface, these curves are called *principal*. Let us investigate the existence of the principal curves. If the principal curves do exist, the equations $\tau^1 = 0$ and $\tau^2 = 0$ are completely integrable, i.e., $\tau^1 \wedge d\tau^1 = 0$ and $\tau^2 \wedge d\tau^2 = 0$. We get

$$(A^2 - C^2)\,\tau_1^2 = CL\tau^1 + CM\tau^2 + A\bar{L}\bar{\tau}^1 + A\bar{M}\bar{\tau}^2\,,$$
$$(A^2 - C^2)\,\bar{\tau}_1^2 = AL\tau^1 + AM\tau^2 + C\bar{L}\bar{\tau}^1 + C\bar{M}\bar{\tau}^2$$

from $(19_2)$ and $(20_2)$, i.e.,

$$\tau^1 \wedge d\tau^1 = -\tau^1 \wedge \tau^2 \wedge \bar{\tau}_1^2 = C(C^2 - A^2)^{-1}\,.\,\tau^1 \wedge \tau^2 \wedge (\bar{L}\bar{\tau}^1 + \bar{M}\bar{\tau}^2)\,,$$
$$\tau^2 \wedge d\tau^2 = \quad \tau^2 \wedge \tau^1 \wedge \tau_1^2 = A(A^2 - C^2)^{-1}\,.\,\tau^2 \wedge \tau^1 \wedge (\bar{L}\bar{\tau}^1 + \bar{M}\bar{\tau}^2)\,.$$

From the existence of the principal curves it follows $L = M = 0$ and $\tau_1^2 = 0$, this being a contradiction.

**Theorem 3.** *A surface in $H^3$ has no principal curves.*

Finally, let us study the geometrical interpretation of the invariants $A$ and $C$. Consider the real representation of the space $H^3$, i.e., the Euclidean space $E^6$ with the complex structure $I : V^6 \to V^6$ ($V^6$ being the underlying vector space of $E^6$) such that $(v_1, v_2) = (Iv_1, Iv_2)$ for each $v_1 . v_2 \in V^6$. Write $\tau^1 = \varphi^1 + i\psi^1$, $\tau^2 = \varphi^2 + i\psi^2$. We have $iv = Iv$ in the considered representation; therefore, we may write

$$dM = \varphi^1 v_1 + \psi^1 I v_1 + \varphi^2 v_2 + \psi^2 I v_2\,,$$
$$dv_1 \equiv A\varphi^1 v_3 + A\psi^1 I v_3\,, \quad dIv_1 \equiv -A\psi^1 v_3 + A\varphi^1 I v_3\,,$$
$$dv_2 \equiv C\varphi^2 v_3 + C\psi^2 I v_3\,, \quad dIv_2 \equiv -C\psi^2 v_3 + C\varphi^2 I v_3$$
$$(\text{mod } v_1, v_2, Iv_1, Iv_2)\,.$$

In a fixed point $M_0 \in M$, let us choose a unit tangent vector $v$ and a curve $\gamma = \gamma(s)$ on $M$ such that $s$ is its arc, $\gamma(s_0) = M_0$ and $d\gamma(s_0)/ds = v$. Denote by $\tilde{v}$ the orthogonal projection of the vector $d^2\gamma(s_0)/ds^2$ into the normal plane (spanned by the vectors $v_3, Iv_3$) of the manifold $M$ at $M_0$. If

$$v = \sin\gamma \sin\alpha\,.\,v_1 + \sin\gamma\cos\alpha\,.\,Iv_1 + \cos\gamma\sin\beta\,.\,v_2 + \cos\gamma\cos\beta\,.\,Iv_2\,,$$

we get

$$\tilde{v} = -(A\sin^2\gamma\cos 2\alpha + C\cos^2\gamma\cos 2\beta)\,v_3 +$$
$$+ (A\sin^2\gamma\sin 2\alpha + C\cos^2\gamma\sin 2\beta)\,Iv_3\,,$$

and the vector $\tilde{v}$ depends on $v$ only. $|\tilde{v}|$ being the length of the vector $\tilde{v}$ and $\varrho = |\tilde{v}|^2$, we have

$$\varrho = A^2\sin^4\gamma + C^2\cos^4\gamma + 2AC\sin^2\gamma\cos^2\gamma\cos 2(\alpha - \beta)\,.$$

Let us look for a vector $v$ for which $|\tilde{v}|$ has an extremal value. We have

$$\frac{\partial \varrho}{\partial \alpha} = -AC \sin^2 2\gamma \,.\, \sin 2(\alpha - \beta)\,, \quad \frac{\partial \varrho}{\partial \beta} = AC \sin^2 \gamma \,.\, \sin 2(\alpha - \beta)\,,$$

$$\frac{\partial \varrho}{\partial \gamma} = 2 \sin 2\gamma \,.\, \big(A^2 \sin^2 \gamma - C^2 \cos^2 \gamma + AC \cos^2 \gamma \,.\, \cos 2(\alpha - \beta) -$$
$$- AC \sin^2 \gamma \,.\, \cos 2(\alpha - \beta)\big)\,.$$

Thus we have $\sin 2\gamma = 0$ or $\sin 2(\alpha - \beta) = 0$. Suppose $\sin 2(\alpha - \beta) = 0$. Then

$$\frac{\partial \varrho}{\partial \gamma} = 2(A - C) \sin 2\gamma \,.\, \big(A \sin^2 \gamma + C \cos^2 \gamma\big)\,,$$

and because of $A > 0$, $C > 0$, we have $A \sin^2 \gamma + C \cos^2 \gamma > 0$ and $\sin 2\gamma = 0$. Thus, we have always $\sin 2\gamma = 0$. For $\sin \gamma = 0$, we get

$$v = \sin \beta \,.\, v_2 + \cos \beta \,.\, Iv_2 = (\sin \beta + i \cos \beta)\, v_2\,, \quad \varrho = C^2\,;$$

for $\cos \gamma = 0$, we have

$$v = \sin \alpha \,.\, v_1 + \cos \alpha \,.\, Iv_1 = (\sin \alpha + i \cos \alpha)\, v_1\,, \quad \varrho = A^2\,.$$

The geometrical interpretation of the invariants is thus sufficiently described.

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