Pavla Vrbová
On local spectral properties of operators in Banach spaces

*Czechoslovak Mathematical Journal*, Vol. 23 (1973), No. 3, 483–492


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ON LOCAL SPECTRAL PROPERTIES OF OPERATORS
IN BANACH SPACES

Pavla Vrbová, Praha

(Received July 12, 1972)

In his paper [5] P. R. Halmos has introduced the notion of capacity in a Banach algebra, particularly in the algebra of operators on a Hilbert space. The notion turns out to be of use in investigations of properties of quasialgebraic elements of a Banach algebra (an element \(a\) in the algebra \(A\) is called quasialgebraic if there exists a sequence \((p_n)\) of monic polynomials, with degree \(d(p_n)\), such that \(|p_n(a)|^{1/d(p_n)} \to 0\) as \(n \to \infty\)). In this note we shall prove some additional properties of quasialgebraic elements.

As regards operators, there arises the natural question how to define a locally quasialgebraic operator \(T\) acting on a Banach space \(X\). F. H. Vasilescu proposed in [8] a definition of to be locally quasialgebraic which relates this notion to spectra of elements \(x \in X\) with respect to \(T\). In the present note we prove the existence of a comparatively large set of elements \(x \in X\) with extremal spectra, namely \(\gamma_T(x) \supset \sigma(T)\) and \(\sigma_T(x) = \sigma(T)\) (See [4]). This result is then used to obtain a proof of the fact that every locally quasialgebraic operator is quasialgebraic.

The paper is divided into two parts. In the first we shall describe some spectral properties of elements of \(X\), the second part is devoted to the notion of capacity. To make the paper self-contained we shall give some proofs included in [5], [7], [8].

1. Let \(X\) be a Banach space, let \(T \in B(X)\) (the algebra of all linear bounded operators from \(X\) to \(X\)) be given. Denote by \(\Omega_T \subset \mathbb{C}\) (the complex plane) the set of all complex \(\lambda\) for which there exists a neighbourhood \(U_\lambda\) with the property that for every open set \(\omega \subset U_\lambda\) the unique holomorphic solution of the equation \((\lambda - T)f(\lambda) \equiv 0\) on \(\omega\) is the function \(f \equiv 0\) in \(\omega\). The set \(\Omega_T\) is open by its definition. The set \(\Omega_T\) is called the set of analytic uniqueness of \(T\) and the set \(S_T = \mathbb{C} \setminus \Omega_T\) is called the analytic residuum of \(T\) [7]. If \(S_T = \emptyset\) we say that \(T\) has the single-valued extension property [2].

Let \(x \in X\) be given. Take a complex number such that the equation \((\lambda - T)f(\lambda) = x\) has a holomorphic solution in some neighbourhood of this number; denote
by $\delta_T(x)$ the set of all complex numbers with this property. Further, set $\gamma_T(x) = C \setminus \delta_T(x)$, $\varrho_T(x) = \delta_T(x) \cap \Omega_T$ and $\sigma_T(x) = C \setminus \varrho_T(x) = \gamma_T(x) \cup S_T$. Clearly $\gamma_T(x) \subset \sigma_T(x) \subset \sigma(T)$. It is an easy application of the open mapping theorem that $\bigcup_{x \in X} \sigma_T(x) = \sigma(T)$ (or equivalently $\bigcap_{x \in X} \varrho_T(x) = \varrho(T)$). See [7].

Remark 1. The set $S_T$ is closed, $S_T = (\text{Int } S_T)^-$. Moreover, $S_T$ is the closure of the set of all points $\lambda$ such that there exists an $h_\lambda \neq 0$ with the property $\lambda - T) h_\lambda = 0$, $\gamma_T(h_\lambda) = \emptyset$.

Proof. The set $S_T$ consists of all $\lambda$ for which in every neighbourhood $U_\lambda$ there is an open set $\omega \subset U_\lambda$ and a non-zero holomorphic function $f$ satisfying

$$(\mu - T) f(\mu) = 0$$

in $\omega$.

It follows that $\omega \subset \text{Int } S_T$ and consequently $S_T = (\text{Int } S_T)^-$. Further, if $\lambda \in S_T$, we may suppose that $\omega$ is such that $f$ has only a finite number of zeros in $\omega$. Take arbitrary $\mu \in \omega$ such that $f(\mu) \neq 0$. Then $\gamma_T(f(\mu)) = \emptyset$. Indeed, $(\mu - T) f(\mu) = 0$ so that the function $h(\xi) = (\xi - \mu)^{-1} f(\mu)$ is holomorphic for $\xi \neq \mu$ and

$$(\xi - T) h(\xi) = (\mu - \omega) h(\xi) + (\xi - \mu) h(\xi) = f(\mu)$$

hence $\gamma_T(f(\mu)) \subset (\mu)$. On the other hand, the function $g(\xi) = (\mu - \xi)^{-1} (f(\xi) - f(\mu))$ for $\xi \neq \mu$, $g(\mu) = -f'(\mu)$ is holomorphic in $\omega$ and

$$(\xi - T) g(\xi) = (\xi - T) (\mu - \xi)^{-1} f(\xi) - (\mu - T) (\mu - \xi)^{-1} f(\mu) + f(\mu) = f(\mu)$$

for $\xi \neq \mu$ and by continuity everywhere in $\omega$. Thus $\mu \in \delta_T(f(\mu))$ as well and $\gamma_T(f(\mu)) = \emptyset$. To prove the assertion it is sufficient to prove that every $\lambda$ with the property $(\lambda - T) h = 0$ for some $h \neq 0$ and $\gamma_T(h) = \emptyset$ belongs to $S_T$. Let $f$ be holomorphic in a neighbourhood $U_\lambda$ of $\lambda$ such that $h = (\mu - T) f(\mu)$ in $U_\lambda$. The function $g(\mu) = (\lambda - T) f(\mu)$ is non-zero in $U_\lambda$. Moreover

$$(\mu - T) g(\mu) = (\lambda - T) (\mu - T) f(\mu) = (\lambda - T) h = 0$$

in $U_\lambda$. Thus $U_\lambda \subset \text{Int } S_T$. The proof is complete.

1.1. Let $x \in X$ be given. The following conditions are equivalent:

1° $\lambda_1, \ldots, \lambda_m \in \delta_T(x)$,

2° there exist a positive $k$ and a sequence $(x_n) \subset X$ such that

$$(\lambda_1 - T) \ldots (\lambda_m - T) x_{n+1} = x_n, \quad |x_n| \leq k^n \text{ for } n \geq 1 \text{ and }$$

$$(\lambda_1 - T) \ldots (\lambda_m - T) x_1 = x.$$

Proof. Suppose that 1° is satisfied. Then there exist an $\varepsilon > 0$ and a holomorphic

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function $f$ defined in $M = \bigcup_{i=1}^{m} D(\lambda_i, 2\varepsilon)$ ($D(\lambda_i, 2\varepsilon)$ is the open disk with center $\lambda_i$
and radius $2\varepsilon$; the sets $D(\lambda_i, 2\varepsilon)$ are supposed to be pairwise disjoint) such that
$(\mu - T)f(\mu) = x$ in $M$. Form an infinite sequence $(\mu_i)$ setting $\mu_n = \lambda_k$ for every natural $n$ in the
form $m\ell + k$, $1 \leq k \leq m$. Define by induction the sequence of functions $g_i$ on $M$ as follows:

$$
g_1 = f, \quad g_n(\mu) = (\mu - \mu_{n-1})^{-1}(g_{n-1}(\mu_{n-1}) - g_{n-1}(\mu)) \quad \text{for} \quad \mu \neq \mu_{n-1},
$$

$$
g_n(\mu_{n-1}) = -g_{n-1}(\mu_{n-1}) \quad (n \geq 2).
$$

Then all $g_n$ are holomorphic in $M$. Using continuity of $g_{n+1}$ and the equality

$$(\mu - T)g_{n+1}(\mu) = (\mu - T)(\mu - \mu_n)^{-1}g_n(\mu_n) - (\mu - T)(\mu - \mu_n)^{-1}g_n(\mu) =
$$

$$
g_n(\mu_n) + (\mu_n - T)(\mu_n - \mu_n)^{-1}g_n(\mu_n) - (\mu_n - T)(\mu_n - \mu_n)^{-1}g_n(\mu)
$$

for $\mu \neq \mu_n$ we obtain by induction $(\mu - T)g_{n+1}(\mu) = g_n(\mu_n)$ in $M$, $n \geq 1$.

Denote by $(g_n)_k$ the $k$-th coefficient of the expansion of $g_n$ in a power series in $D(\lambda_i, \varepsilon)$, i.e. $g_n(\mu) = \sum_{k=0}^{\infty} (g_n)_k (\mu - \lambda_i)^k$ in $D(\lambda_i, \varepsilon)$. We shall prove by induction that

$$
|(g_n)_k| \leq K(1/\varepsilon)^{k+n}
$$

for some $K$. Indeed, we have $\limsup_{k \to \infty} |(f)_k|^{1/k} \leq (2\varepsilon)^{-1}$ and consequently $|(g_1)_k| \leq K\varepsilon^{-k-1}$ for suitable $K$ and $1 \leq i \leq m$. Further, we have, for

$$
\lambda_i = \mu_n, \quad g_{n+1}(\mu) = -\sum_{k=0}^{\infty} (g_n)_k (\mu - \lambda_i)^k \quad \text{in} \quad D(\lambda_i, 2\varepsilon).
$$

If $\lambda_i = \mu_n$, then

$$
-g_{n+1}(\mu) = (\mu - \mu_n)^{-1}\left(\sum_{k=0}^{\infty} (g_n)_k (\mu - \lambda_i)^k - g_n(\mu_n)\right) =
$$

$$
= (\sum_{k=0}^{\infty} (-1)^k (\mu - \lambda_i)^k (\lambda_i - \mu_n)^{-k-1} (\sum_{k=0}^{\infty} (g_n)_k (\mu - \lambda_i)^k - g_n(\mu_n)) =
$$

$$
= \sum_{k=0}^{\infty} ((-1)^k (\lambda_i - \mu_n)^{-k-1} (g_n)_0 + \ldots + (\lambda_i - \mu_n)^{-1} (g_n)_k -
$$

$$
- (-1)^k (\lambda_i - \mu_n)^{-k-1} g_n(\mu_n)(\mu - \lambda_i)^k \quad \text{in} \quad D(\lambda_i, 2\varepsilon).
$$

By the induction assumption we have

$$
|(g_{n+1})_k| = |(g_n)_{k+1}| \leq K\varepsilon^{-k-n-1} \quad \text{for} \quad \lambda_i = \mu_n
$$

and

$$
|(g_{n+1})_k| \leq K(\sum_{j=0}^{k} (2\varepsilon)^{-k-1+j} \varepsilon^{-j-n} + (2\varepsilon)^{-k-1} \varepsilon^{-n}) =
$$

$$
= K\varepsilon^{-k-n-1}(\sum_{j=0}^{k} (2^{-k-1+j} + 2^{-k-1}) = K\varepsilon^{-k-n-1} \quad \text{for} \quad \lambda_i = \mu_n.
$$

It follows that $|g_n(\mu_n)| \leq K\varepsilon^{-n}$ for $n \geq 1$. Set $x_n = g_m(\mu_m)$. 485
Then we have $|x_n| \leq K \epsilon^{-m,n}$ and the equality $(\mu - T) g_{n+1}(\mu) = g_n(\mu_n)$ implies $(\lambda_1 - T) \ldots (\lambda_m - T) x_{n+1} = x_n$ for $n \geq 1$, $(\lambda_1 - T) \ldots (\lambda_m - T) x_1 = x$. This proves $2^\circ$.

To prove the implication $2^\circ \to 1^\circ$ it suffices to show that the function $f_\lambda(\lambda) = \sum_{n=0}^\infty (-1)^n \prod_{j \neq i, j=1}^m (\lambda_j - T)x_{n+1}(\lambda - \lambda_i)^n$ satisfies $(\lambda - T) f_\lambda(\lambda) = x$ in $D(\lambda_i, \epsilon_i)$, where $\epsilon_i = (\prod_{j \neq i, j=1}^m |(\lambda_j - T)| k)^{-1}$. Indeed, we have

$$(\lambda - T) f_\lambda(\lambda) = (\lambda - \lambda_i) f_\lambda(\lambda) + (\lambda_i - T) f_\lambda(\lambda) =$$

$$\sum_{n=0}^\infty (-1)^n \prod_{j \neq i, j=1}^m (\lambda_j - T)x_{n+1}(\lambda - \lambda_i)^n + x +$$

$$+ \sum_{n=1}^\infty (-1)^n \prod_{j \neq i, j=1}^m (\lambda_j - T)^n x_n(\lambda - \lambda_i)^n = x.$$

This completes the proof.

1.2. Suppose that $0$ does not belong to $S_T$. If $TX = X$ then $T^{-1} \in B(X)$.

Proof. Denote by $X_1$ the unit ball of $X$. According to the open mapping theorem $X_1 \subset kTX_1$ for some positive $k$. Given an $x \in X$ we can define a sequence of elements $x_n \in X$ such that $Tx_1 = x$, $Tx_{n+1} = x_n$ and $|x_n| \leq k^n|x|$ for $n \geq 1$. It follows that $0 \in \delta_T(x)$ by 1.1. Together with the assumption $0 \notin S_T$ it gives $0 \in g_T(x)$ for every $x$ and consequently, $0 \in g(T)$. The proof is complete.

1.3. $\sigma(T) = S_T \cup \{\lambda : (\lambda - T) B(X) = B(X)\} = S_T \cup \{\lambda : T X = X\}$

Proof. Clearly $S_T \cup \{\lambda : (\lambda - T) X = X\} \subset S_T \cup \{\lambda : (\lambda - T) B(X) = B(X)\} \subset \subset \sigma(T)$. To prove the assertion it suffices to show that

$$\sigma(T) \subset S_T \cup \{\lambda : (\lambda - T) X = X\}$$

or equivalently,

$$\Omega_T \cap \{\lambda : (\lambda - T) X = X\} \subset g(T).$$

If $\lambda \in \Omega_T$ then $0 \in \Omega_{\lambda - T}$ and it follows that $(\lambda - T)^{-1} \in B(X)$ by 1.2. The proof is complete.

The following corollary is related to [1].

1.4. Let $H$ be a Hilbert space, let $T \in B(H)$. If $T^*$ has the single-valued extension property then $\sigma(T) = \sigma_\delta(T)$ (the approximate point spectrum).

Proof. Take an $S \in B(H)$. A complex number $\lambda$ does not belong to $\sigma_\delta(S)$ if and only if $\inf_{|x| = 1} |(\lambda - S)x| > 0$. The last inequality is equivalent to the fact that $(\lambda - S)$
is one-to-one and its range is closed. This is, in its turn, equivalent to the existence of an operator $R$ such that $R(\lambda - S) = I$. Indeed, denote by $P$ the projection on $(\lambda - S)H$ then $(\lambda - S)(\lambda - S)^{-1}P(\lambda - S) = I$. On the other hand if $R(\lambda - S) = I$ then $|x| \leq |R| |(\lambda - S)x|$ for every $x \in H$ and $\inf \{ \lambda : |(\lambda - S)x| \geq |R|^{-1} \}$. Hence, $\sigma_\delta(S) = \{ \lambda : B(H)(\lambda - S) \neq B(H) \}$. Since $T^*$ has the single-valued extension property we obtain by 1.3 $\sigma(T) = \sigma(T^*)^* = \{ \lambda : (\lambda - T^*)B(H) \neq B(H) \}^* = \{ \lambda : B(H)(\lambda - T) \neq B(H) \} = \sigma_a(T)$.

1.5. Theorem. Let $T \in B(X)$. Then each of the sets \{ $x : \gamma_T(x) \supset \text{bd} \sigma(T)$, $x : \sigma_T(x) = \sigma(T)$ \} is nonmeagre (i.e. it is not of the first category) in $X$.

Proof. Denote by $L_1$ (resp. $L_2$) a countable dense subset of $\text{bd} \sigma(T)$ (resp. $\sigma(T) \setminus S_T$). Since $L_1 \subset \text{bd} \sigma(T)$ it follows $(\lambda - T)X \neq X$ for every $\lambda \in L_1$. Moreover, the set $(\lambda - T)X$ is of the first category in $X$ by the open mapping theorem and so is the set $\bigcup_{\lambda \in L_1} (\lambda - T)X$. Taking an $x \notin \bigcup_{\lambda \in L_1} (\lambda - T)X$ we obtain $L_1 \subset \gamma_T(x)$ and consequently, $\text{bd} \sigma(T) = L_1 \subset \gamma_T(x)$.

To prove the theorem it remains to prove that the set \{ $x : \sigma_T(x) = \sigma(T)$ \} is nonmeagre in $X$. If $\sigma(T) = S_T$ then $\sigma_T(x) = S_T$ by its definition so that $\sigma_T(x) = \sigma(T)$ for each $x \in X$. Hence assume that $\sigma(T) \setminus S_T \neq \emptyset$. Given $\lambda \in \sigma(T) \setminus S_T$ we have $(\lambda - T)X \neq X$ by 1.2. Again, the set $\bigcup_{\lambda \in L_1} (\lambda - T)X$ is of the first category in $X$ and we have $\gamma_T(x) \supset L_2$ for every $x \notin \bigcup_{\lambda \in L_1} (\lambda - T)X$. It follows that $\sigma_T(x) = \gamma_T(x) \cup \bigcup_{\lambda \in L_2} (\lambda - T)X \neq \text{bd} \sigma(T)$. The proof is complete.

Remark. On the other hand, let us mention that the set of those elements $x \in X$ for which $\sigma_T(x) \neq \sigma(T)$ may consist only of zero element. Take, for instance, the example presented by R. C. Sine [6] (See also [4]), i.e. the isometric shift in $l_2$, $T(\xi_1, \xi_2, \ldots) = (0, \xi_1, \xi_2, \ldots)$. The operator $T$ has the single-valued extension property and $\sigma_T(x) = \sigma(T)$ for every $0 \neq x \in l_2$. Indeed, since $T$ is isometric, we have $|((\lambda - T)x)| \geq |(1 - |\lambda|)| |x| > 0$ for $|\lambda| < 1$ and $0 \neq x \in l_2$. It follows that $T$ has no eigenvalue in the open unit disc and clearly it has the single-valued extension property. Further, suppose that a complex number $\lambda_0$, $|\lambda_0| < 1$ belongs to $\sigma_T(x)$ for some $x = (x_n)_{n=1}^\infty \in l_2$. Then $x \in (\lambda - T)X$ and $x \in \text{Ker} (\lambda^* - T^*)$ for $\lambda$ in a neighbourhood $U_0$ of $\lambda_0$. The only eigenvector corresponding to $T^*$ and to $\lambda^*$ is $x_\lambda = \lambda^* x_1$ so that $\sum_{n=1}^\infty \lambda_n x_n = 0$ for all $\lambda \in U_0$. It follows that the power series $\sum_{n=1}^\infty \mu_n x_n$ converges also for all $|\mu| < |\lambda_0|$ and $\sum_{n=1}^\infty \mu_n x_n = 0$. Thus $x_n = 0$ for $n \geq 1$ and $x = 0$. Since $|T| = 1$ it follows that $\sigma(T) = D(0, 1)^+ = \sigma_T(x)$ for every $x \in l_2$.

The operator $T^*$ has not the single-valued extension property and $\gamma_{T^*}(x)$ is empty or $\gamma_{T^*}(x) \subset \text{bd} \sigma(T^*) = \{ \lambda : |\lambda| = 1 \}$ for every $x \in l_2$. [10].

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The following theorem (with the exception of the proof) is included in [9]. We shall give now an elementary proof.

Denote by $\mathcal{F}(T)$ the set of all functions holomorphic in a neighbourhood of $\sigma(T)$.

1.6. Let $f \in \mathcal{F}(T)$ which is non-constant in each component of definition. Then

1° $f(\gamma_T(x)) = \gamma_{f(T)}(x)$ for every $x \in X$

2° $f(\sigma_T(x)) = \sigma_{f(T)}(x)$ for every $x \in X$

3° $S_{f(T)} = f(S_T)$.

Proof. Let $x \in X$ be given. The inclusion $f(\gamma_T(x)) \subset \gamma_{f(T)}(x)$ being trivial for $\gamma_T(x) = \emptyset$ we shall suppose that $\gamma_T(x) \neq \emptyset$. Given $\lambda$, let $g \in \mathcal{F}(T)$ be such that $f(z) - f(\lambda) = (z - \lambda)g(z)$. Suppose that $f(\lambda) \in \delta_{f(T)}(x)$. Let $(x_n)$ be a sequence corresponding to $f(\lambda)$ and to the operator $f(T)$ by 1.1. Set $y_n = g^n(T)x_n$ for $n \geq 1$. Then the sequence $(y_n)$ satisfies

$$(T - \lambda) y_{n+1} = (T - \lambda) g^{n+1}(T) x_{n+1} = g^n(T)(T - \lambda) g(T) x_{n+1} = g^n(T) y_n = y_n,$$

$$(T - \lambda) y_1 = (T - \lambda) g(T) x_1 = x.$$ 

It follows that $\lambda \in \delta_T(x)$ and consequently, $f(\gamma_T(x)) \subset \gamma_{f(T)}(x)$.

On the other hand, take $\lambda \in \gamma_{f(T)}(x)$. The function $f$ is non-constant so that $\lambda - f$ has only a finite number of zeros in some neighbourhood of $\sigma(T)$ so there exists a $g \in \mathcal{F}(T)$, $g(z) \neq 0$ for all $z$ and $\lambda - f(z) = (\lambda_1 - z) \ldots (\lambda_m - z) g(z)$. Suppose that all $\lambda_i \in \delta_T(x)$. By 1.1 we can choose elements $x_n$ such that

$$(\lambda_1 - T) \ldots (\lambda_m - T) x_{n+1} = x_n,$$

$|x_n| \leq k^n$ for $n \geq 1$ and some positive $k$, $(\lambda_1 - T) \ldots (\lambda_m - T) x_1 = x$. Put

$$y_n = (1/g)^n(T) x_n.$$ 

Then $(\lambda - f(T)) y_{n+1} = y_n$ and $|y_n| \leq ((1/g)(T)) k^n$ for $n \geq 1$, $(\lambda - f(T)) y_1 = x$. Thus $\lambda \in \delta_{f(T)}(x)$ which is a contradiction. Hence $\lambda_i \in \gamma_f(x)$ for some $i$ and

$$\lambda = f(\lambda_i) \in f(\gamma_T(x)).$$ 

This completes the proof of 1°.

Now we shall prove $f(S_T) = S_{f(T)}$. Take $\lambda$ such that $(\lambda - T) h = 0$ with a suitable $h \neq 0$, $\gamma_T(h) = 0$. Then also $(f(\lambda) - f(T)) h = 0$ and by 1° $\gamma_{f(T)}(h) = 0$ as well. This proves the inclusion $f(S_T) \subset S_{f(T)}$. Further, let $\lambda$ be such that $(\lambda - f(T)) h = 0$ with an $h \neq 0$, $\gamma_{f(T)}(h) = 0$. To prove the inclusion $S_{f(T)} \subset f(S_T)$ it is sufficient to show that $f(\lambda) \in f(S_T)$ for each such $\lambda$.

By 1° $\gamma_T(h) = 0$ as well. Let $\lambda - f(z) = (z - \lambda_1) \ldots (z - \lambda_m) g(z)$ with $g \in \mathcal{F}(T)$
and \( g \) has no zeros. We have \( 0 = (\lambda - f(T)) h = (T - \lambda_1) \ldots (T - \lambda_m) g(T) h \). Suppose that all \( \lambda_i \in \Omega_T \). Let \( j \) be the greatest index such that
\[
(T - \lambda_j) \ldots (T - \lambda_m) g(T) h = 0.
\]
Denote by \( y = (T - \lambda_{j+1}) \ldots (T - \lambda_n) g(T) h \neq 0 \). If \( j = n \) set \( y = g(T) h \).
Since \( 1/g \in \mathcal{F}(T) \) we have \( g(T) h \neq 0 \).
Since \( (\lambda_j - T) y = 0 \) we have \( (\lambda - T)(\lambda - \lambda_j)^{-1} y = y \) for \( \lambda \neq \lambda_j \). Since \( \lambda_j \in \Omega_T \) it follows that \( \gamma_T(y) = (\lambda_j) \). On the other hand, \( \emptyset = \gamma_T(h) \supseteq \gamma_T(y) \) which is a contradiction. It follows that at least one \( \lambda_i \in S_T \) and \( \lambda = f(\lambda_i) = f(S_T) \). We proved 3°.

We obtain condition 2° as follows:
\[
\sigma_{f(T)}(x) = \gamma_{f(T)}(x) \cup S_{f(T)} = f(\gamma_T(x)) \cup f(S_T) = f(\gamma_T(x) \cup S_T) = f(\sigma_T(x)).
\]

2. Denote by \( P_{n}^1 \) the set of all monic polynomials (i.e. the leading coefficient is 1) of degree \( n \), by \( P^1 \) the set of all monic polynomials. By \( d(p) \) we shall denote the degree of \( p \in P^1 \).

**Definition.** Let \( A \) be a Banach algebra with a unit over the complex field, let \( a \in A \) be given. Denote by \( \text{cap}_n a = \inf_{p \in P_{n}^1} |p(a)| \) and by \( \text{cap} a = \lim_{n} (\text{cap}_n a)^{1/n} \). The number \( \text{cap} a \) is called the capacity of \( a \).

The norm in \( A \) is submultiplicative so that \( \text{cap}_{n+a} a \leq \text{cap}_n a \cdot \text{cap}_a a \) and thus \( \lim_{n} (\text{cap}_n a)^{1/n} \) exists and \( \text{cap} a = \inf_{n} (\text{cap}_n a)^{1/n} \).

Let \( K \) be a compact set in the complex plane. Consider the algebra \( C(K) \) with norm \( |f|_K = \sup_{z \in K} |f(z)| \). Then the capacity of the identical function \( a(\lambda) = \lambda \), i.e. \( \lim_{n \to \infty} (\inf_{p \in P_{n}^1} |p|_K)^{1/n} \) is known as the Čebyšev constant of the set \( K \) and we shall denote it by \( \text{cap} K \). Set \( \text{cap} \emptyset = 0 \).

The present definition of capacity differs slightly from that given in the paper of P. R. Halmos [5]. We shall give now a shorter proof of some results included in [5].

**2.1.** \( \text{cap} a = \inf_{p \in P^1} |p(a)|^{1/d(p)} = \inf_{p \in P^1} |p(a)|^{1/d(p)} \).

**Proof.** The relation \( \text{cap} a = \inf_{p \in P^1} |p(a)|^{1/d(p)} \) follows immediately from the definition of capacity. Further, the inequality \( |p(a)|_\sigma \leq |p(a)| \) gives
\[
\inf_{p \in P^1} |p(a)|^{1/d(p)} \leq \inf_{p \in P^1} |p(a)|^{1/d(p)}.
\]

On the other hand, we have
\[
\text{cap} a \leq \inf_{p \in P_{nk}^1} |p(a)|^{1/nk} \leq \inf_{p \in P_{n}^1} |p^k(a)|^{1/nk}
\]

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for every natural \( k, n \) and this implies

\[
\cap a \leq \inf_{k, n} \inf_{p \in \mathbb{P}_n} |p^k(a)|^{1/nk} = \inf_{n} \inf_{p \in \mathbb{P}_n} |p^k(a)|^{1/n} = \inf_{n} |p(a)|^{1/n} = \inf_{p \in \mathbb{P}_1} |p(a)|^{1/d(p)}.
\]

This completes the proof.

2.2.

\[
\cap a = \cap \sigma(a).
\]

Proof. According to 2.1 and to the spectral mapping theorem

\[
\cap \sigma(a) = \inf_{p \in \mathbb{P}_1} |p|^{1/d(p)} = \inf_{p \in \mathbb{P}_1} |p(a)|^{1/d(p)} = \cap a.
\]

Definition. An element \( a \in A \) is said to be algebraic if \( p(a) = 0 \) for some \( p \in \mathbb{P}_1 \). An element \( a \) is said to be quasialgebraic if there exists a sequence of polynomials \( p_n \in \mathbb{P}_1 \) such that \( |p_n(a)|^{1/d(p_n)} \to 0 \). It follows from 2.1 that

2.3. The following assertions are equivalent:

1° there exists a sequence of monic polynomials \( (p_n) \) such that \( |p_n(a)|^{1/d(p_n)} \to 0 \),

2° \( \cap a = 0 \),

3° there exists a sequence of monic polynomials \( (p_n) \) such that \( |p_n(a)|^{1/d(p_n)} \to 0 \).

2.4. Let \( a \in A \) be quasialgebraic, let \( f \) be a function holomorphic in a neighbourhood of \( a \). Then \( f(a) \) is quasialgebraic.

Proof. According to 2.2 we have \( \cap \sigma(a) = 0 \). According to a classical theorem [3] (285–291) there exist monic polynomials \( t_n \) of degree \( n \) with roots in \( \sigma(a) \) such that \( |t_n|^{1/n} = \cap \sigma(a) = 0 \). Write \( t_n(z) = (z - \lambda_1) \cdots (z - \lambda_n) \).

Define functions \( f_n^i \) (\( i = 1, 2, \ldots, n \)) by the following two postulates: \( f_n^i(z) = (f(z) - f(\lambda_i))^{-1} \) for \( z \neq \lambda_i \), \( f_n^i(z) = f'(\lambda_i) \) for \( z = \lambda_i \). All functions \( f_n^i \) are holomorphic in the same neighbourhood of \( \sigma(a) \) as \( f \) and we have \( |f_n^i| \leq \leq 2|f'| \sigma(a) \). Set \( f_n(z) = \prod_{i=1}^{n} f_n^i(z) \). Then we have \( |f_n| \sigma(a) \leq \leq [2|f'| \sigma(a) \ d(\sigma(a))]^n \).

Further, take the polynomial \( q_n(z) = \prod_{i=1}^{n} (z - f(\lambda_i)) \). The order of \( q_n \) is \( n \) and \( q_n(f(z)) = f_n(z) t_n(z) \) for \( z \neq \lambda_i \), by continuity equality holds everywhere. Further, the following estimate is true

\[
|q_n|^{1/n} = |q_n \circ f|^{1/n} = |f_n^i|^{1/n} \leq |f_n|^{1/n} |t_n|^{1/n} \leq 2|f'| \sigma(a) \ d(\sigma(a)) \cdot |t_n|^{1/n}.
\]

The last term tends to zero as \( n \to \infty \) and this implies that \( 0 = \cap f(\sigma(a)) = \cap \sigma(f(a)) = \cap f(a) \).

Let \( X \) be a Banach space. 2.3 provides different methods how to define locally quasialgebraic operators in the algebra \( B(X) \). In his paper [8] F. H. Vasilescu took
the local version of $3^\circ$ which turned out to be the local version of $2^\circ$. We shall begin with some definitions: Given an $x \in X$, denote by $r(T, x) = \limsup_{n} |T^*x|^1/n$. Clearly $r(T, x)$ is the radius of the minimal disk outside of which $x = (\mu - T) f_\alpha(\mu)$ for a holomorphic function $f_\alpha$.

Denote by $\text{cap}_n(T, x) = \inf_{p \in P^n} r(p(T), x)$ and by $\text{cap}(T, x) = \inf_{p \in P^1} r(p(T), x)^{1/d(p)}$.

2.5. $\text{cap} \gamma_T(x) \leq \text{cap} (T, x) \leq \text{cap} \sigma_T(x)$ for every $x \in X$.

Proof. Clearly $\gamma_T(x) \subset \{ \lambda : |\lambda| \leq r(p(T), x) \} \subset \sigma_T(x)$. It follows $|a|_{\gamma_T(x)} \leq r(p(T), x) \leq |a|_{\sigma_T(x)}$ (recall that $a$ is the identical function). From 1.6 it follows that $|p|_{\gamma_T(x)} = |a|_{\gamma_T(x)} \leq r(p(T), x) \leq |a|_{\sigma_T(x)} = |p|_{\sigma_T(x)}$ for each $p \in P^1$ and consequently $\gamma_T(x) \subset \text{cap} (T, x) \subset \text{cap} \sigma_T(x)$. The proof is complete.

Now, we shall give an example of an operator for which the inequalities in the preceding proposition are strict.

Let $S$ be the left shift on the space $l_1$. Denote by $e_i = (\xi_1^i, \xi_2^i, \ldots)$ such that $\xi^i_k = \delta_{ik}$. Then $\sum l_1 = l_1 l_1 e_i = \sum l_1 l_1 e_i$. Denote by $X$ the Banach space of all bounded sequences $x = (x_n)$, $x_n \in l_1$ with the supremum norm, $|x| = \sup |x_n|$. Define the operator $T \in B(X)$ by the formula $(Tx)_n = Sx_n$. Since $x_\lambda = \sum x_\lambda/1 e_n$ satisfies $Sx_\lambda = \lambda x_\lambda$ for $|\lambda| < 1$ if follows that $|\lambda| < 1$ is an eigenvalue of $T, |T| = 1$ so that $\sigma(T) = D(0, 1)$, $S_T = D(0, 1)$. Indeed, $(\mu - S) \sum e_n = 0$ in $D(0, 1)$ and $\sum e_n = 1/(1 - |\mu|)$. Thus $D(0, 1) \subset S_T \subset \sigma(T) = D(0, 1)$.

Take a sequence $x_\lambda$ of complex numbers such that $(x_\lambda, x_\lambda, \ldots) = D(0, 1)$ for some $0 < \lambda < 1$. Form a sequence $x_0 = (x_0) \subset X$ such that $Sx_0 = \lambda x_0$, $|x_0| = 1$. We shall show that $0 = \text{cap} \gamma_T(x_0) \subset \text{cap} (T, x_0) = \text{cap} D(0, \varepsilon) \subset \text{cap} \sigma_T(x_0) = \text{cap} D(0, 1)$. Clearly $\sigma_T(x_0) = S_T = D(0, 1)$.

Observe that $|p^k(T) x_0|^{1/k} = (\sup |p^k(S)| x_0)^{1/k} = (\sup |p^k(\lambda)|)^{1/k} = \sup |p(\lambda)| = 1/(1 - |\mu|)$ and $r(p(T), x_0) = |p|_{D(0, \varepsilon)}$. From the last equality it follows that $\text{cap} (T, x_0) = \text{cap} D(0, \varepsilon)$.

Take $0 < \delta < 1$ arbitrary. Set $f_\lambda(\mu) = \sum (\mu^\lambda - \lambda^\mu)(\mu - \lambda)^{-1} e_n^{n+1}$ for $\mu \neq \lambda, \mu \in D(0, \delta), \lambda \in D(0, 1)$. The function $f_\lambda$ can be extended to an analytic function $f_\lambda$ in $D(0, 1)$. It follows $D(0, \delta) \subset \sigma_T(x_0)$. Now, take $\varepsilon' < \varepsilon < 1$ and define $g_\lambda(\mu) = (\mu - \lambda)^{-1} x_0$ for $|\mu| > \varepsilon$. Then $(\mu - S) g_\lambda(\mu) = x_0$ and $|g_\lambda(\mu)| \leq 1/(\varepsilon' - \varepsilon)$ for $|\mu| > \varepsilon'$. Thus $\delta_T(x_0) = \{ |\lambda| \geq \varepsilon' \}$ and $\gamma_T(x_0) = \emptyset$.

Definition. An operator $T \in B(X)$ is said to be locally quasialgebraic if for every $x \in X$ there exists a sequence of monic polynomials $p_n$ such that $r(p_n(T), x)^{1/d(p_n)} \to 0$. 491
2.6. Every locally quasialgebraic operator is quasialgebraic.

Proof. Let \( T \in B(X) \) be locally quasialgebraic. By definition of cap \((T, x)\) and by 2.6 it follows that \( \text{cap} \gamma_T(x) = 0 \) for every \( x \in X \). Choose an \( x \neq 0 \) such that \( \gamma_T(x) \supseteq \text{bd} \sigma(T) \) by 1.4. Then \( \text{cap} \sigma(T) = \text{cap} \text{bd} \sigma(T) \leq \text{cap} \gamma_T(x) = 0 \). This completes the proof.

References


Author's address: 115 67 Praha 1, Žitná 25, ČSSR (Matematický ústav ČSAV v Praze).