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STRUCTURE OF MAXIMAL SPECTRAL SPACES OF GENERALIZED SCALAR OPERATORS

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Let X be a Banach space, let $B(X)$ be the algebra of all linear bounded operators from X to X. Denote by C^∞ the algebra of all infinite times differentiable complex functions defined on the complex plane $\mathbf{C}$ with the topology of uniform convergence of every derivate on each compact set in $\mathbf{C}$, i.e. with the topology generated by a family of pseudonorm $|\varphi|_{K,m} = \max_{|p| \leqq m} \sup_{z \in K} |D^p f(z)|$, where K is arbitrary compact set, m a non-negative integer, $p = (p_1, p_2)$, $|p| = p_1 + p_2$ and

$$D^p f = \frac{\partial^{|p|} f}{\partial z_1^{p_1} \partial z_2^{p_2}} \quad (z = z_1 + i z_2).$$

A spectral distribution is a multiplicative vector-valued distribution $\mathcal{U}: C^\infty \to B(X)$ for which $\mathcal{U}(1) = I$. Denote by a the function $a(\lambda) = \lambda$ for $\lambda \in \mathbf{C}$. An operator $T \in B(X)$ is said to be generalized scalar if there exists a spectral distribution $\mathcal{U}$ such that $\mathcal{U}(a) = T$. This class of operators was introduced in paper [2] of C. FOIAŞ. In this paper the author proved a theorem describing structure of certain class of invariant subspaces of generalized scalar operators, so called maximal spectral spaces [1]. It is the purpose of this note to give another characterization of these invariant spaces which is an analogy of the finite dimensional case. The presented methods are closely related to [3].

First we shall recall some definitions and known results concerning generalized scalar operators included in [2], [1].

Let $T \in B(X)$ be a generalized scalar operator. Denote by $\varrho_T(x)$ the set of all complex numbers λ for which there exists a holomorphic solution of the equation

$$(\xi - T) f(\xi) = x$$

in some neighbourhood of λ. Set $\sigma_T(x) = \mathbf{C} \setminus \varrho_T(x)$. We have $\sigma_T(\mathcal{U}(\varphi) x) \subset \operatorname{supp} \varphi$ for $x \in X$, $\varphi \in C^\infty$. If $F = F^- \subset \mathbf{C}$ then $X_T(F) = \{x : \sigma_T(x) \subseteq F\}$ is a closed hyperinvariant subspace with respect to T. The space $X_T(F)$ is maximal spectral, i.e. it has the following property: if $Z \subset X$ is a closed subspace invariant with respect to T

and $\sigma(T \mid Z) \subset \sigma(T \mid X_T(F))$ then $Z \subset X_T(F)$, and $\sigma(T \mid X_T(F)) \subseteq F$. Further, if we denote by X^G the linear subspace spanned by all elements of the form $\mathcal{U}(\varphi)\, x$ with $\varphi \in C^\infty$ such that $\operatorname{supp} \varphi \subset G$, G open, $x \in X$ arbitrary, then $X_T(F) = \bigcap_{G \supset F} X^G$. It follows that $x \in X_T(F)$ if and only if $\mathcal{U}(\varphi)\, x = 0$ for all $\varphi \in C^\infty$ such that $\operatorname{supp} \varphi \cap F = \emptyset$.

Now, we shall begin with certain considerations concerning the decomposition of the unit on compact sets in the space C^∞.

If $\varphi \in C^\infty$ we shall denote by $\int \varphi(z)\, dz = \iint \varphi(x + iy)\, dx\, dy$ $(z = x + iy)$.

1.1. *Let m be a nonnegative integer. Then there exists $k_m > 1$ with the following property: Let K be arbitrary compact set, let $(G_i)_{i=1,2}$ be a covering of K, i.e. $K \subset \operatorname{Int} G_1 \cup \operatorname{Int} G_2$ such that $K \setminus G_i \neq \emptyset$ and $d(G_i) \leqq 1$ $(i = 1, 2)$. Denote by $\varepsilon = \min_{i=1,2} d(G_i, K \setminus G_{3-i}) > 0$. Then there exist $\varphi_1, \varphi_2 \in C^\infty$ with the properties: $\varphi_1 + \varphi_2 = 1$ in a neighbourhood of the set K, $\operatorname{supp} \varphi_i \subseteq G_i$ and $\sup |D^j \varphi_i| \leqq$ $\leqq k_m \varepsilon^{-4-|j|} \max_{i=1,2} d(G_i)^4$ for $i = 1, 2$ and $|j| \leqq m$.*

Proof. Let $i = 1, 2$. There exist compact sets K_i such that $D(K_i, \varepsilon/3) \subset G_i$ and $K \subset K_1 \cup K_2$ ($D(M, \delta)$ is the set of all λ for which $d(\lambda, M) < \delta$). Take a nonnegative function $\varphi_0 \in C^\infty$, $\int \varphi_0 = 1$, $\operatorname{supp} \varphi_0 = D(0, 1)^-$. Let u_i be the characteristic function of the set $D(K_i, \varepsilon/4)$. Define functions

$$\psi_i(\mu) = \int u_i(\mu - 12^{-1}\varepsilon\lambda)\, \varphi_0(\lambda)\, d\lambda = (\varepsilon/12)^{-2} \int u_i(\lambda)\, \varphi_0(12(\mu - \lambda)/\varepsilon)\, d\lambda\,.$$

It follows that $0 \leqq \psi_i \leqq 1$, $\psi_i(\mu) = 1$ for $\mu \in D(K_i, \varepsilon/6)$ and $\operatorname{supp} \psi_i \subset D(K_i, \varepsilon/3) \subset$ $\subset G_i$. Further,

$$D^j \psi_i = (\varepsilon/12)^{-2} \int u_i(\lambda)\, D^j(\varphi_0(12(\mu - \lambda)/\varepsilon))\, d\lambda$$

so we obtain $\sup |D^j \psi_i| \leqq (\varepsilon/12)^{-2-|j|} \sup |D^j \varphi_0|^2\, d(G_i)^2$. Set $\varphi_1 = \psi_1$, $\varphi_2 =$ $= \psi_2(1 - \psi_1)$. Since $K \subset K_1 \cup K_2$, we have $\varphi_1 + \varphi_2 = 1$ in a neighbourhood of K. Clearly $\operatorname{supp} \varphi_i \subset \operatorname{supp} \psi_i \subset G_i$. The Leibniz formula yields the following estimate for φ_2 and $j = (j_1, j_2)$, $|j| \leqq m$

$$\sup |D^j \varphi_2| = \sup \left| \sum_{k,l=0}^{j_1,j_2} \binom{j_1}{k} \binom{j_2}{l} D^{(k,l)} \psi_2 D^{(j_1-k, j_2-l)}(1 - \psi_1) \right| \leqq$$

$$\leqq 1 + \sum_{(k,l) \neq (j_1,j_2)} \binom{j_1}{k} \binom{j_2}{l} \sup |D^{(k,l)} \psi_2| \sup |D^{(j_1-k, j_2-l)} \psi_1| \leqq$$

$$\leqq 1 + \sum_{(k,l) \neq (j_1,j_2)} \binom{j_1}{k} \binom{j_2}{l} (\varepsilon/12)^{-2-k-l} \max_{|j| \leqq m} \sup |D^j \varphi_0|^4\, d(G_2)^2\,.$$

$$.\, (\varepsilon/12)^{-2-j_1-j_2+k+l}\, d(G_1)^2 \leqq 1 + (\varepsilon/12)^{-4-|j|} \max_{|j| \leqq m} \sup |D^j \varphi_0|^4\,.$$

$$.\, \max_{i=1,2} d(G_i)^4\, (2^{|j|} - 1)\,.$$

Since $(\varepsilon/12)^{4+|j|} \leqq d(G_i)^4$ we have

$$\sup |D^j \varphi_2| \leqq (\varepsilon/12)^{-4-|j|} \max_{|j| \leqq m} \sup |D^j \varphi_0|^4 \max_{i=1,2} d(G_i)^4 \,.\, 2^{|j|} .$$

Set

$$k_m = 2^m \,.\, 12^{4+m} \max_{|j| \leqq m} \sup |D^j \varphi_0|^4 .$$

Then we obtain

$$\sup |D^j \varphi_2| \leqq k_m \varepsilon^{-4-|j|} \max_{i=1,2} d(G_i)^4 \quad \text{for} \quad |j| \leqq m .$$

It is easy to verify that we have the same estimate for the function φ_1 as well.

1.2. *Let T be a generalized scalar operator. Then there exists a natural number p such that $X_T(F) = \bigcap_{\lambda \notin F} (\lambda - T)^p X$ for every closed set F.*

Proof. The operator T possesses a spectral distribution $\mathcal{U}$, so there exists a $K > 0$, a natural number m and a compact neighbourhood U of the set $\sigma(T)$ such that $|\mathcal{U}(\varphi)| \leqq K|\varphi|_{U,m}$ for every $\varphi \in C^\infty$. Let $(\sqrt{2})^{-1} < c < 1$ be given. Denote by $b = (\sqrt{2} \,.\, c - 1)(4 + 2\sqrt{2})^{-1} < 1$. Choose a p natural such that $c^{p-m} < (2k_m(b^{-1}c)^4)^{-1}$, where k_m is a constant corresponding to m by 1.1.

To prove the inclusion $\bigcap_{\lambda \notin F} (\lambda - T)^p X \subset X_T(F) \, (F = F^-)$ it suffices to prove that $\mathcal{U}(\varphi)\, x = 0$ for every $\varphi \in C^\infty$ with the support disjoint with F and for every $x \in \bigcap_{\lambda \notin F} (\lambda - T)^p X$. It is easy to see that it suffices to consider only φ with supports included in arbitrary isosceles rectangular triangle D with the hypotenuse $d < 1$, $D \cap F = \emptyset$. Now, consider a required triple x, φ, D. Cover D by two similar triangles with hypotenuses dc so that the number ε corresponding by 1.1 to this covering be equal db. Hence, by 1.1 there exists a function φ_1 with support in one of the smaller triangles such that $\sup |D^j \varphi_1| \leqq k_m(db)^{-4-|j|} (dc)^4 = k_m(db)^{-|j|} (b^{-1}c)^4$ and $|\mathcal{U}(\varphi)\, x| \leqq 2|\mathcal{U}(\varphi\varphi_1)\, x|$. We can define, by induction, a sequence of triangles D_n and sequence of function φ_n with properties: $d(D_n) = dc^n$, $\operatorname{supp} \varphi_n \subset D_n$, $\sup |D^j \varphi_n| \leqq k_m(dc^{n-1}b)^{-4-|j|} (dc^n)^4 = k_m(b^{-1}c)^4 (dbc^{n-1})^{-|j|}$ for $|j| \leqq m$ and $|\mathcal{U}(\varphi)\, x| \leqq 2^n |\mathcal{U}(\varphi\varphi_1 \ldots \varphi_n)\, x|$.

By induction we obtain

$$\sup |D^j \varphi_1 \ldots \varphi_n| \leqq (k_m(b^{-1}c)^4)^n (db)^{-|j|} \left(1 + \frac{1}{c} + \ldots + \frac{1}{c^{n-1}}\right)^{|j|}$$

for $|j| \leqq m$ and all n. Indeed, applying the induction hypothesis, we obtain

$$\sup |D^j \varphi_1 \ldots \varphi_n \varphi_{n+1}| \leqq \sum_{k,l=0}^{j_1,j_2} \binom{j_1}{k} \binom{j_2}{l} \sup |D^{(k,l)} \varphi_1 \ldots \varphi_n| \sup |D^{(j_1-k,j_2-l)} \varphi_{n+1}| \leqq$$

$$\leqq (k_m(b^{-1}c)^4)^n \, k_m(b^{-1}c)^4 \sum_{k,l=0}^{j_1,j_2} \binom{j_1}{k} \binom{j_2}{l} (db)^{-k-l} .$$

$$\cdot\left(1+\frac{1}{c}+\ldots+\frac{1}{c^{n-1}}\right)^{k+l}(dbc^n)^{-j_1-j_2+k+l}=$$
$$=\left(k_m(b^{-1}c)^4\right)^{n+1}(db)^{-|j|}\left(1+\frac{1}{c}+\ldots+\frac{1}{c^n}\right)^{|j|}.$$

Denote by $\lambda_0 = \bigcap_n \operatorname{supp} \varphi\varphi_1 \ldots \varphi_n$. The number λ_0 belongs to D so there exists a $y \in X$ such that $x = (\lambda_0 - T)^p y$. Then we have

$$|\mathscr{U}(\varphi)\,x| \leqq 2^n|\mathscr{U}(\varphi\varphi_1 \ldots \varphi_n)\,x| = 2^n|\mathscr{U}(\varphi \ldots \varphi_n(\lambda_0 - a)^p)\,y| \leqq$$
$$\leqq 2^n|\mathscr{U}(\varphi)\,y|\,K \max_{|j|\leqq m} \sup_{D_n} \Big|\sum_{k,l=0}^{j_1,j_2} \binom{j_1}{k}\binom{j_2}{l} D^{(k,l)}(\varphi_1 \ldots \varphi_n)\, D^{(j_1-k,j_2-l)}(\lambda_0 - a)^p\Big| \leqq$$
$$\leqq M_m\left(2k_m(b^{-1}c)^4 c^p\right)^n\left(1+\frac{1}{c}+\ldots+\frac{1}{c^n}\right)^m.$$

The last term tends to zero according to definition of p. We proved the inclusion $\bigcap_{\lambda \notin F} (\lambda - T)^p X \subset X_T(F)$. The relation $\sigma(T \mid X_T(F)) \subset F$ implies the reverse inclusion. The proof is complete.

An immediate consequence of 1.2 is the following collorary related to [3], [4], [5].

1.3. *Let S be a linear transformation (without assumption of continuity) commuting with a scalar generalized operator T.*

Then $SX_T(F) \subset X_T(F)$ for $F = F^-$.

In view of the preceding collorary we can reformulate the Theorem 3.5 in [5] as follows:

1.4. *Let T be a generalized scalar operator in a Banach space X which has no critical eigenvalue (i.e. range $(\lambda - T)X$ has finite codimension for every eigenvalue λ). Let S be a linear transformation commuting with T.*

Then S is continuous.

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